The Origin of the Chinese Abacus

By Yoemon YAMAZAKI

Chapter I
Introductory

It is generally admitted to-day that, the earliest people in the Orient who enjoyed the benefits of the abacus, to say nothing of the Hindoos who only had access to the blessings of its most primitive type, were the Chinese.

However, as to when or how the abacus originated in China, there is no universally accepted theory, the Han period theory, the Sung period theory, the Yüan period theory, the Ming period theory and the Ch'ing period theory being held by several writers.

1. The Han period theory

Those who hold the Han period theory are ;

a. Shingen TAKEDA 武田眞元, the author of the *Shingen Sampō* 眞元算法 (1844) ;

b. Goichi SAWADA 澤田吾一, the author of the *Nippon Sūgaku-shi-kōwa* 日本數學史講話 (Discourse on the History of Japanese Mathematics) (1927) ;

c. Shūichirō YOSHIOKA 吉岡修一郎, the author of the *Kan-ji to Kan no Soroban* 漢字と漢の算盤 (The Han Ideographs and the Abacus of the Han Dynasty) in the *Jōkyū Sūgaku* 上級數學 (The Advanced Mathematics) 7–11 (1936) ;

d. Yoemon YAMAZAKI 山崎與右衛門, the present writer, in his *Chūgoku Soroban Oboegaki* 中國算盤覺書 (Notes on the Chinese Abacus) (1954), also definitely holds that the Chinese, as early as in the Han period, were already in the possession of the abacus.

The ground on which the advocates of the Han theory stand is the account in the *Shu-shu-chi-i* 數術記遺.

This work was compiled toward the end of the Han period by Hsü Yo 徐岳 and annotated by Chên Luan 甄鸞 of the Latter Chou period. Hsü Yo is said to have studied calendars under Liu Hung 劉洪 during the reign of Emperor Ling-ti 靈帝 (168–188) of the Han dynasty. The text of this book offers the following passage as the answer of Master Tien-mu 天目 when Liu Hung visited him and inquired of him as to the subject of mathematics.

先生曰，隷首注術，乃有多種，乃餘遺忘，記憶數事而已，其一積算，其一太乙，其一兩儀，其一三才，其一五行，其一八卦，其一九宮，其一運算，其一了知，其一成數，其一把頭，其一龜算，其一珠算，其一計算

" My master said " As for the methods which Li Shou 隷首 expounded, they are so many that I have forgotten them. I now remember only a few. One of them is T'ai-i 太乙 ; one of them, Liang-i 兩儀 ; one of them, San-ts'ai 三才 ; one of them, Wu-hsing 五行 ; one of them, Pa-kua 八卦 ; one of them, Chiu-kung 九宮 ; one of them, Yün-suan 運算 ; one of them, Liao-chih 了知 ; one of them, Chêng-shu 成數 ; one of them, Pa-t'ou 把頭 ; one of them, Kuei-suan 龜算 ; one of them, Chu-suan 珠算 (abacus calculation) ; and one of them Chi-suan 計算."
Thus he mentions 14 methods of calculation. Among them is one entitled Chu-suan 珠算 (abacus calculation) and the Notes on the text alone reads 珠算，控帶四時，經緯三才 (Abacus calculation controls all four seasons and covers the three orders—heaven, earth and men.)

This would make it impossible to understand the construction of the calculative instrument. But Chên Luan 甄鸞 comments on it as follows:—

刻版爲三分，其上下二分，以停游珠，中間一分，以定算位，位各五珠，上一珠與下四珠色別，其上別色之珠當(五)，其下四珠，珠各當一，至下四珠所領，故云控帶四時，其游珠於三方之中，故云經緯三才也

" A wooden board is carved into three compartments ; the two compartments, the top and the bottom contain mobile beads, and the middle compartment is used for determining the order of the numbers. Five beads can be put in each column ; the beads in the top compartment and those in the bottom compartment are differently colored. The top bead differently colored designates five and each bead in the bottom represents one. The middle compartment is full when occupied by four beads which represent one. Hence the expression 控帶四時—literally, to control all four seasons. And the mobile beads are placed in the middle of the three compartments ; hence the expression 經緯三才—literally, to cover the three orders—heaven, earth and man."

From this explanation in the book, an abacus of a proto-type of the present-day abacus is observed.

(Note. Chên Luan 甄鸞, in the middle of the 6th century, annotated *Chiu-*

chang 九章, *Sun-tzŭ* 孫子, *Wu-ts'ao* 五曹, *Chang Ch'iu-chien* 張丘建, *Hsia-hou Yang* 夏侯陽, *Chou-pi* 周髀 and *Wu-ching* 五經 *Shu-shu-chi-i* 數術記遺. Only on the strength of these *Notes* of his, the works came to be first recognized as authentic writings.)

Should this be accepted without reserve, there could be no doubt whatever in dating the abacus from the Han period. However, there is a view which questions the genuineness of the book *Shu-shu-chi-i* 數術記遺.

Dr. Yoshio MIKAMI 三上義夫, in his *Shina-shisō* 支那思想 (Chinese Thought) Kagaku, Sūgaku, 1934, remarks " The text by Hsü Yo 徐岳 is an extremely brief one ; and, seeing that the CHÊN Luan 甄鸞 goes too far to defend the text, it might be suspected that CHÊN Luan wrote this book under Hsü Yo's name. That CHÊN Luan contributed much toward the defence of Buddhism is recorded in the *Fo-tsu-li-tai-t'ung-tsai* 佛祖歷代通載 (The General History of Buddhism in China, by Nien-ch'ang 念常 in Yüan dynasty). Existence of such a Buddhist phrase as *kṣana* 刹那 (moment) in the book would suggest its relation to Buddhism." There exists a theory which rejects the *Shu-shu-chi-i* as a counterfeit. However, it would not be much mistaken to date the book somewhere around CHÊN Luan's time. Even rejected as a counterfeit and a later work, the work representing various methods of denoting numbers would be worth serious consideration.

Dr. Akira HIRAYAMA 平山諦, in his remarks on *Warizan-sho* 割算書 (Book on Division), 1954, says : " That the book *Shu-shu-chi-i* 數術記遺 itself is not free from suspicion." I here by put on record by reprinting a passage which Mr. Hisashi SASSA 佐々久, Chief of the Miyagi Prefectural Library, has kindly forwarded me, —a passage quoted from under the item Tzŭ in the *Ch'in-ting-ssŭ-k'u-ch'üan-shu-chien-ming-mu-lu* 欽定四庫全書簡明目錄 :—

術數記遺 (*Shu-shu-chi-i*, in one volume)

舊本題漢徐岳撰北周甄鸞注，隋書不著錄，序中所言姓名時代多與史傳牴牾，註亦無所發明，疑爲僞作，殆因唐代算學書肆有，此書遂襲其名，而依託歟，流傳已久，姑錄以備一家焉

" According to the title, the original is by Hsü Yo of the Han period, and the Notes by CHÊN Luan 甄鸞 of the Pei-Chou 北周 period. The *Sui-shu* 隋書 fails to include it. Statement in its *Preface* contradicts most histories and biographies so far as the names and times are concerned. Neither do the *Notes* offer any original interpretation. It is suspected as a counterfeit book. Even since the T'ang period this book has been included in the lists of the publishers until the work has assumed this title, probably due to solicitation. At any rate it has been circulated for such a long time. For this reason, it is recorded here for the reference of that school." At the beginning, the book is represented as the *Shu-shu-chi-i* 術數記遺, but most copies extant to-day are entitled ' *Shu-shu-*

chi-i 數術記遺'. The statement as to the names and times does not agree with those given in the histories and biographies. Besides, the *Notes* do not give any original interpretation, either. So it may be suspected as a counterfeit work. Among the mathematical works imported into Japan during the Nara 奈良 period, an account of the *Shu-shu-chi-i* 數術記遺 occurs, but the book mentioned there may not be of the same one as the present. The *Ssŭ-k'u-ch'üan-shu-tsung-mu* 四庫全書總目 dwells at considerable length on proving it as a counterfeit.

In China, the *Suan-hsüeh-tz'ŭ-tien* 算學辭典 (1939) compiled by TUAN Yü-huo 段育華 and CHOU Yüan-jui 周元瑞 contains the following passage.

數術記遺一卷，卷首題漢徐岳撰，北周甄鸞注，係宋鮑澣之從道藏中錄出，隋書經籍志無此書，舊唐書經籍志，唐書藝文志，但載其名，四庫全書總目，以爲唐人所作，嫁名於徐岳著，今刊於算經十書中

"The *Shu-shu-chi-i* 數術記遺, complete in one volume. The title is chosen by Hsü Yo 徐岳, and annotated by CHÊN Luan 甄鸞 of the Pei-Chou 北周 period, but PAO Han-chih 鮑澣之 of the Sung period includes it in his *Tao-tsang* 道藏; the *Sui-shu-ching-chi-chih* 隋書經籍志 does not cite this book, while both the *Chiu-t'ang-shu-ching-chi-chih* 舊唐書經籍志 and the *T'ang-shu-i-wên-chih* 唐書藝文志 mention this title. The *Ssŭ-k'u-ch'üan-shu-tsung-mu* 四庫全書總目 calls it a work by some T'ang author, but falsely published under Hsü Yo's name. It is now included in the *Suan-ching-shih-shu* 算經十書". It is thus suspected in China.

On the other hand, Dr. Yoshio MIKAMI 三上義夫 in his *Shina Sūgaku no Tokushoku* 支那數學の特色 (Characteristics of Chinese Mathematics), *Tōyō Gakuhō* 東洋學報, Vol. XV, No. 4, 1926, remarks: "The method of abacus calculation represented in the *Shu-shu-chi-i* 數術記遺 is of the same principle as those practised in Greece and Rome, and was probably introduced from the West." Furthermore, Dr. Kinnosuke OGURA 小倉金之助, in his *Sūgaku-shi Kenkyū* 數學史研究, Pt. 1, (1935), says "The view that holds the abacus was first introduced from Rome to China (subsequently from there to Japan), (though lacking evidential material) is probably an adequate inference." Mr. LI Yen 李儼, in his *Chung-suan-shih-lun-ts'ung* 中算史論叢 (A Paper on the History of Chinese Mathematics), Pt. 4 (1955), states:—

但徐岳數術記遺所稱之珠算，非卽柯尙遷所示之算盤，因柯書明記，"初定算盤圖式"，則其時上距發明，自尙不遠，而徐岳數術記遺所載，或僅與西洋人所用者相類

"Only what Hsü Yo 徐岳 calls chu-suan 珠算 (bead-calculation) in the *Shu-shu-chi-i* 數術記遺 is not what Ko Shang-chien 柯尙遷 indicates as suan-p'an 算盤 (abacus). For Ko definitely writes 初定算盤圖式 (In the beginning of this book is shown a figure of an abacus). So the time was not much prior to the invention of the abacus. And what Hsü Yo gives in the *Shu-shu-chi-i* was probably somewhat similar to one the Westerners used."

The present writer Yoemon YAMAZAKI, in my *Chūgoku Soroban Oboegaki*

中國算盤覺書 (Notes on the Chinese Abacus), writes as follows : " In view of the fact that the abacus appeared 30 or 40 centuries B.C. in an early civilized community in Mesopotamia, and introduced through ancient civilized countries such as Egypt, Greece and Rome, it enjoyed extensive use in the various mediaeval countries, the view that holds that China was the sole inventor of the abacus could not be accepted unconditionally.

" Studying the construction of the abacus represented in the said *Shu-shu-chi-i* 數術記遺, this writer is convinced that the Chinese abacus was either introduced from Rome or was closely related with the Roman.

" It is said that the introduction of the abacus to China dates from the Han period. I wish to explain here my position based on these two points of view ". I said, " In the closest correlation of the economic structure which is the foundation of society, the ideology built upon this foundation, and the skill which is the bulwark of industrial production which constitutes the nucleus of the economic construction of community, how the abacus grew and developed through the respective historic periods and what was the type of development it assumed—from these points of view I have conducted my investigation, and have come to the following conclusion. The grooved Roman abacus was first introduced to China, and then, after much remodelling in Oriental fashion, to Japan. The abacus imported to China from Rome was extremely crude and much credit is due to the Chinese people who remodelled the simple abacus into the Chinese style abacus—the direct mother type of the Japanese abacus. I would by no means grudge due praise to their meritorious achievement." This summarizes what I then said in my " Notes on the Chinese Abacus."

2. The Sung period theory

Those who favor this theory are :

a. Tsune HOSHINO 星野恒 : *Soroban no Denrai* 算盤の傳來 (The Introduction of the Abacus), 1893, *Shigaku Zasshi* 史學雜誌 4–44.
b. Yoshio MIKAMI 三上義夫 ; *Kuku ni tsukite* 九九に就きて (On the Reckoning-Rhymes), 1921, *Tōyō Gakuhō* 東洋學報, 11–1 ; *Shina Sūgaku no Tokushoku* 支那數學の特色 (The Characteristics of Chinese Mathematics), Tōyō Gakuhō 東洋學報, 15–4 ; *Tōzai Sugaku-shi* 東西數學史 (History of Mathematics in the East and West).
c. D. E. SMITH : History of Mathematics, 1925.
d. CH'IEN Pao-ts'ung 錢寶琮 : *Ku-suan-k'ao-yüan* 古算考源 (Study of Ancient Mathematics), 1930.
e. Fusakichi ORIHARA 織原房吉 : *Soroban no Enkaku* そろばんの沿革 (Development of the Abacus). *Chūgai Shōgyō Shimpō* 中外商業新報 (Commercial Daily News) March 15 and 17, 1931, issues.

f. Sasaki Endō 遠藤佐々喜: *Soroban Raireki-kō* 算盤來歷考, 同補遺 (Study of the Introduction of the Abacus, and Supplement.) 1931–36, *Shigaku* 史學, June, 1931 ; July 1936.

The foundation of the Sung period theory consists in the following facts :—

(i) In the *Suan-hsüeh-yüan-liu* 算學源流 (Origin of Mathematics) in the *Suan-fa-t'ung-tsung* 算法統宗 (Preface, 1592) by Ch'êng Ta-i (Ssŭ-fu) 程大位(思甫) are mentioned the titles of the mathematical books previous to this book ; and among those published since Yüan-fêng 元豐 (1078–1085), Shao-hsing 紹興 (1131–1162), and Ch'un-hsi 淳熙 (1178–1189) in Sung dynasty, there occur such titles as *P'an-chu-chi* 盤珠集 and *Tsou-p'an-chi* 走盤集. Therefore, the book is considered to represent the time of the appearance of the abacus by (a) and (b) among others.

(ii) Under the item *Suan-fa-t'ung-tsung* 算法統宗 in the *Ssu-k'u-ch'üan-shu-tsung-mu* 四庫全書總目 (Bk. 101) this passage occurs :—

珠算之名，始見甄鸞周髀註，則北齊已有之，然所說與今頗異，梅文鼎謂起於元末明初，不知宋人三珠戲語已有算盤珠之說，則是法盛行於宋矣

" The name chu-suan 珠算—abacus mathematics—is for the first time introduced, with the notes by Chên Luan Chou-pi 甄鸞周髀. According to it, the system already existed in the North-Ch'i 北齊 period. However, it was considerably different from the modern. Mei Wên-ting 梅文鼎 says it first rose in the earlier days of the Ming period. He did not know the existence of the San-chu-hsi-yü 三珠戲語 and the existence of the Suan-p'an-chu 算盤珠 (Abacus beads). The practice became extremely popular in the Sung period." Some stand on this ground ; (a) for instance.

(iii) On account of the explanatory notes on the words Chung, suan-p'an-chih-chung 算盤之中 (Centre of the abacus) used in the *Suan-ching* 算經 by Hsieh Ch'a-wei 謝察徵 of the Sung period, some ascribe the abacus to the Sung period ; (a) for instance.

(iv) Those who do so on account of the appearance of the Kuei-ch'u-ko-chüeh 歸除歌訣 (Division-Rhymes) ; for instance, (e), though he assigns it to the end of the Sung period or at the beginning of the Ming period, or (d).

(v) Those who hold that the Chinese abacus was invented probably by adopting the Hindoo-style abacus and adjusting it, referring to the abacus of the Han period. (f) though he puts it at the end of the Sung period or the beginning of the Yüan period.

(vi) Smith says : " No literature definitely describing an abacus previous to 1175 is found." It is not clear whether he estimated 1175 from (i). As to the expression San-chu-hsi-yü 三珠戲語 under (ii) refers to the following passage under the item Ching-chu 井珠 in the *Cho-kêng-lu* 輟耕錄 (Bk. 29), by T'AO Tsung-i (Nan-ts'un) 陶宗儀 (南村) of the last days of the Yüan period.

凡納婢僕，初來時日擂盤珠，不撥自動，稍久日算盤珠，言撥之則動，旣久日佛頂珠，言終日凝然雖撥亦不動，此雖俗諺實切事情

"When a maid or a man-servant newly hired comes for the first time, she is called 'lei-p'an-chu 擂盤珠'—a mortar bead. When ordered, she works of her own accord without being beaten. After a while, she is called 'suan-p'an-chu 算盤珠'—an abacus bead. Only when ordered and beaten, she will work. After a long while, she is called 'fo-ting-chu 佛頂珠'—a sour face bead. When ordered, she will stand idle all day long and even when beaten, she will not work. Though somewhat vulgar figures, these may serve to explain the situation realistically."

This being a book written in the last days of the Yüan period, it does not prove the existence of the expression "San-chu-hsi-yü" in the Sung period which preceded the Yüan.

Dr. MIKAMI, in his "*Soroban no Gogen ni kansuru Shin-setsu no Hihan* ソロバンの語源に關する新說の批判 (Criticism of the New Theory concerning the Etymology of the Abacus)", *Tokyo Butsuri Gakkō Zasshi* 東京物理學校雜誌, No. 541, 542, 1936, says: "I have roughly read through the whole work in 30 Bks., but have not been able to find the term Tsou-p'an-chu 走盤珠."

3. The Yüan period theory

a. CH'IEN Ta-hsin 錢大昕: *Shih-chia-ch'i-yang-hsin-lu* 十駕齋養新錄 (*Preface*, 1799).

b. Tōsato KONDŌ 近藤遠里: *Sūgaku Yawa* 數學夜話 (Nightly Talks on Mathematics), 1751.

c. Ujikazu FURUKAWA 古川氏一: *Sanwa Zuihitsu* 算話隨筆 (Stories on Mathematics).

d. Tomokiyo OYAMADA 小山田與淸: *Matsu-no-ya Hikki* 松屋筆記 (Notes of Matsu-no-ya) (1783–1847).

e. Seijitsu SATŌ 佐藤誠實: *Kokushi Dai-jiten* 國史大辭典 (Dictionary of Japanese History), 1908.

f. Keinosuke TAKAI 高井計之助: *Soroban Zatsuwa* 算盤雜話 (Chat on the Abacus). *Tōkyō Kōen-Dōkō-kai Kaihō* 東京講演同好會 會報 No. 267, Oct. 15, 1931.

g. Kunizō ARIMOTO 有本邦造: *Shina ni okeru Shuzan no Kigen* 支那に於ける珠算の起源 (Origin of Abacus Mathematics in China). *Yamaguchi Kōshō Tōa-Keizai Kenkyu* 山口高商東亞經濟研究 15–3.

h. Kiyoshi YABUUCHI 藪内淸: *Shina Sūgaku-shi* 支那數學史 (History of Chinese Mathematics), 1944.

i. Matsusaburō FUJIWARA 藤原松三郎: *Nihon Sūgaku Shiyō* 日本數學史要 (Outline of Japanese Mathematics), 1952.

j. Akira HIRAYAMA 平山諦: *Warizan-sho Kaisetsu* 割算書解說 (Exposition of Division Books), 1952.

The above are principal works on the subject.

As for the foundation of the theory :—

i Some argue from the aforesaid term suan-p'an-chu 算盤珠 represented in the *Cho-kêng-lu* 輟耕錄 1366; for instance, a, d, e, g, i, :

ii Men (like j) contend from the Kuei-ch'u-ko-chüeh 歸除歌訣 (Division-Rhymes) in the *Suan-hsüeh-chi-mêng* 算學啓蒙, 1299, and Fei-kuei 飛歸 (Special division-rhymes for 2 place numbers), in the *Shou-shih-li-chieh-fa-li-ch'êng* 授時曆捷法立成, 1298.

The view of b, c, f, and h are only too vague. Among these, c and f assign it to the last part of the Yüan period and the beginning of the Ming period; as stated previously, CH'IEN Pao-tsung 錢寶琮 places it between the Sung and Yüan periods; Fusakichi ORIHARA 織原房吉 and Sasaki ENDŌ 遠藤佐々喜 who contend for the last part of the Sung dynasty and the beginning of the Yüan are here classified for convenience' sake, under the Sung period theory.

4. The Ming period theory

MEI Wên-ting 梅文鼎: *Ku-suan-ch'i-k'ao* 古算器考 (Study of Ancient Calculating Contrivances) (1693) stands for this, saying as follows:

然則今用珠盤起於何時，日古書散亡苦無明據，然以愚度之亦起明初耳，何以知之，日歸除歌訣最爲簡妙，此珠盤所恃以行也，然九章比類所載句長而澁，蓋即是時所創，後人踵事增華，乃更簡快耳，是書爲錢塘呉信民作，其年月可考而知則珠盤之來固自不遠

"As to when the Chu-p'an 珠盤 (abacus) which we now use came into being, it is regrettable that all the ancient records being scattered and lost, no definite evidence is to be found. The present writer MEI Wên-ting is of the opinion that it originated only in the earliest days of the Ming period. Why is this known? Because the Kuei-ch'u-ko-chüeh 歸除歌訣 (the Division-Rhymes) is the simplest and cleverest invention upon which chu-p'an 珠盤 operations depend. However, the division rhymes given in the *Chiu-chang-hsiang-chu-pi-lei* 九章詳註比類 are too long and difficult. Those created at this time retain the same substance, but have been refined and simplified by posterity. The author of this book is WU Ching (Hsin-min) 呉敬(信民) of Ch'ien-t'ang 錢塘. If the book could be dated, he believes, the introduction of the abacus could not be very far from it."

Thus he identifies the appearance of the Kuei-ch'u-ko-chüeh 歸除歌訣 with that of the abacus. However, as fully discussed later, it is admitted that the oldest kuei-chu-ko-chüeh had been printed about the 11th or 12th century in the *Chih-nan-*

suan-fa 指南算法 therefore, if MEI Wên-ting's method of argument be followed, the origin of the abacus should have to go back as far as that.

As has been roughly discussed in the foregoing, as to the advent of the abacus, the newest date assigned is the Ming period, the rest being the Han, Sung and Yüan period theories. Of all these theories, the most important issues are :—

a. Should the appearance of the Kuei-ch'u-ko-chûeh 歸除歌訣 (Division-Rhymes) be considered the origin of the abacus?

b. Should my view based on the resemblance between the Roman abacus and the one represented in the *Shu-shu-chi-i* 數術記遺 be confirmed in the point of calculation?

c. Did the *P'an-chu-chi* 盤珠集 and the *Tsou-p'an-chi* 走盤集 really represent the abacus books? Can these books be identified with the origin of the abacus?

d. Is it adequate to suppose the advent of the abacus as substitute for calculation-blocks? In th present paper, I desire to analyse these items chiefly from the she standpoint of calculation methods, and thereby surmise the date of the advent of the abacus in China. For this reason, the terms empolyed for the abacus given in the *Suan-ching* 算經 by HSIEH Ch'a-wei 謝察徵 of the Sung period, the poem on the abacus in the writings of Master Ching-hsiu 靜修先生, 1270, and the San-chu-hsi-yü 三珠戲語 in the *Cho-kêng-lu* 輟耕錄 1366, should better be utilized as material for demonstrating the conditions of its diffusion rather than its origin. This being the case, I have discussed them here only briefly.

Chapter II

The Theory of Identifying the Advent of the Kuei-ch'u-ko-chüeh 歸除歌訣 with that of the Abacus

This view of identifying the date of the advent of the Kuei-ch'u-ko-chüeh 歸除歌訣 with that of the advent of the abacus came into being as the advocates considered the following facts ;—

a. That, since the middle of the Ming period the abacus, replacing the previous use of calculation-blocks, was rapidly and extensively used ;

b. That for a long time since then, the abacus was extensively used as a common calculating contrivance ;

c. That it seemed, consequently, as if the Kuei-ch'u-ko-chüeh-method 歸除法 —that of calculating by reciting the division-rhymes—were the sole division method for the abacus ;

d. That the Kuei-ch'u-ko-chüeh method 歸除法 suits the abacus so perfectly that it was thought to have been invented for the abacus ;

e. That, as the abacus with two beads above the bar, each of which representing 5, is indispensable to the Ch'ung-kuei-fa 撞歸法. For example : 13÷18 ; 24÷29.

In such cases where the top numbers of the divisor and the dividend are similar and the divisor is greater than the dividend, a special set of division-rhymes were adopted to obtain the quotient. An example is Chien-i-wu-t'ou-tso-chiu-i 見一無頭作九一 (See-one-no-head-make-nine-add-one-down).

This was called the Ch'ung-kuei 撞歸 method.

The abacus is in closest relations with the Kuei-ch'u-fa 歸除法 (division-rhymes).

As calculations conducted by means of suan-mu 算木 (Reckoning-blocks) were considered those to be conducted by the method represented in the *Sun-tzŭ-suan-ching* 孫子算經 which is to be discussed later, the kuei-ch'u division rhymes were invented over against such inconvenient calculations, and the abacus was contrived in order to be applied to the reckoning-rhymes and thereby it replaced the calculating-blocks.

Now, Dr. Mikami in his "*Characteristics of Chinese Mathematics*", criticizes this as follows :—" The use of the abacus in China must surely have dated from after the *Shu-shu-chi-i*. The date is not clear, but it probably dates from the Sung period. From the fact that some mathematical books written during the Yüan and Ming periods carry division-rhymes, Mei Wên-ting of the Ch'ing period, argues that abacus calculation was practised at that time. This would sound quite reasonable, but even in block-calculations the division rhymes might be used. And where division is explained by the use of the division rhymes, some may be considered to refer to block-calculation instead of abacus calculation. Therefore, Mei Wên-ting's view should not be relied upon as the only theory ; however, there being some other evidence, it may be said that the abacus most probably dates approximately from the Sung and Yüan periods."

In the following, I also desire throughly to analyse this theory as to (a) how the Kuei-ch'u-fa came into being and (b) how it was developed and to prove the inadequacy of indentifying the advent of the Kuei-ch'u-ko-chüeh with the date of that of the abacus.

1. The ancient calculative methods. (The division method prior to the Kuei-ch'u-fa 歸除法 ; division accompanied by division-rhymes)

In investigating how the Kuei-ch'u-fa came into being, it would be necessary, first of all, to find out what sort of calculative methods had existed prior to the advent of the Kuei-ch'u-fa.

The oldest Chinese book that records the calculative method of multiplication is the *Sun-tzŭ-suan-ching* 孫子算經. It is not clear whether the author, SUN-tzŭ, was a man after the Han period, after the Emperor Ming-ti 明帝 of the Han dynasty. But it seems that this book existed in the last part of the Three Kingdom period (approximately, the middle of the 3rd century).

The book describes the method by the use of calculative blocks, and may be diagrammed as follows:

The method of Multiplication:

As for, 327×6=1962—

1.

(Top row)	327	Put 327 in the top row and 6 in the bottom. Move up 6 in the bottom to the place of hundreds in the top place.
(Middle row)		
(Bottom row)	6	

2.

(Top row)	327	Multiply 3 in the top row by 6 in the bottom, and put 18 the result in the middle. Strike out 3 in the top and move down 6 in the bottom by one place.
(Middle row)	18	
(Bottom row)	6	

3.

(Top row)	27	Multiply 2 in the top row by 6 in the bottom, but 12 the result in the middle row; strike out 2 in the top and move down 6 in the bottom by one place.
(Middle row)	192	
(Bottom row)	6	

4.

(Top row)	7	Multiply 7 in the top by 6 in the bottom, put 42 the result, and strike out 7 in the bottom.
(Middle row)	1962	
(Bottom row)	6	

5.

(Top row)		1962 the result is obtained.
(Middle row)	1962	
(Bottom row)		

The method of division:

As for 6561÷9=729

1.

(Top row)		Put 6561 in the middle row and 9 in the bottom; move up 9 in the bottom to the place of thousands in the middle; as 9
(Middle row)	6561	
(Bottom row)	9	

cannot be obtained in 6, move down 9 in the bottom by one place to the place for hundreds.

2.

(Top row)	7	Put the quotient 7 in the top, subtract from the middle 63, the product of 7 in the top and 9 in the bottom and 9 move down by one place.
(Middle row)	6561	
(Bottom row)	9	

3.

(Top row)	72	Put the quotient 2 in the top, subtract from the middle 18 the product of 2 in the top and 9 in the bottom and move down by one place 9 in the bottom.
(Middle row)	261	
(Bottom row)	9	

4.

(Top row)	729	Put 9 the quotient in the top, and subtract from the middle 81 the product of 9 in the top 9 in the bottom.
(Middle row)	81	
(Bottom row)	9	

5.

(Top row)	729	Strike out 9 in the bottom, and you will obtain 729 the quotient in the top.
(Middle row)		
(Bottom row)		

As to the arrangement of reckoning-blocks. The blocks represented numerals as follows: 𝍠, 𝍡, 𝍢, 𝍣, 𝍤, 𝍥, 𝍦 𝍧, 𝍨, (1, 2, 3, 4, 5, 6, 7, 8, 9.). One-place numbers and hundreds were represented by blocks vertically arranged, while tens and thousands were represented by blocks horizontally arranged. Therefore, 𝍩𝍡𝍫𝍣 stood for 1234.

Over against this calculative method of putting the dividend, divisor, quotient (or product) in the three column the top, middle and bottom, simplification of this method was soon to follow. It is found in the *Hsia-hou-yang-suan-ching* 夏侯陽算經.

2. Simplification of calculations

The *Suan-ching* 算經 by HSIA-HOU Yang 夏侯陽 is said to date from the South-North dynasty period (439–589). Dr. MIKAMI is of the opinion that it is a work later than 535, while LI Yen 李儼 believes that HAN Yen 韓延 of the last part of the 6th century introduced it, that he adapted his own theory to it and that the *Introduction* is by himself.

Of the *Ming-chêng-ch'u-fa* 明乘除法 (Explanations of multiplication and division methods of the Ming period in the *Hsia-hou-yang-suan-ching*, First Volume, gives calculative methods similar to those introduced in the *Sun-tzŭ-suan-ching* 孫子算經. Explanations for some exercise questions in the Middle and Last Volume deserve notice.

As for $12{,}639{,}673\times1.02=128{,}992{,}466.46$—

Direction says : Skip one place and add 2 and the result will be obtained)

$12{,}639{,}673+12{,}639{,}673\times0.02$

To illustrate this point more simply :

For example : 7×13—

(1) Put down 7.

(2) From 13, the multiplicand, 10 is taken off, and 3 and 7 are multiplied and 21 is obtained.

This opration is called " adding 3 " or " attaching 3."

𝍦	7
=|	+21
𝍨|	91

As for $34{,}645{,}734\times0.9=3{,}118{,}116.6$—

Another direction : Or, " by subtracting 1 from the bottom, the result may be obtained."To subtract from the bottom means subtracting the multiplier $\times0.1$ from the bottom of the multiplicand, or subtracting from the first place of the multiplicand multiplied by 10 the multiplier $\times1$. This is a method of multiplication adopted when the multiplier is 8 or 9, so near the number 10. It is called the Sun-chêng 損乘 method.

$(3{,}464{,}573.4-3{,}464{,}573.4\times0.1)$

To simplify this point further :

For example : 8×9—

(1) Put down 8

(2) Multiply 1, the supplementary number of 9 for 10, by 8

Subtract 1 from the bottom

$1\times8=8$

$$\begin{array}{r} 80 \\ -8 \\ \hline 72 \end{array}$$

As for $43{,}675.2\div1.2=36{,}396$—

Direction : First put the original number, halve and divide it by 6.

$(43{,}675.2\div2\div0.6)$

Another direction : Subtract 2 from outside itself.

$43{,}675.2=36{,}396\times1.2$

Therefore :

$(43{,}675.2-36{,}396\times0.2)$

As for $3{,}456\div1.2=2880$—

Direction : Subtract 2.

$3{,}456=2{,}880\times1.2$

$3{,}456-2{,}880\times0.2$

To simplify this further:
As for 24÷12—
(1) Put down 24.
(2) Subtract from the next place 4,—the result of multiplying 2, left over by subtracting 10 from 12, by 2, the number of the top place.—

=	||||		24
−	||||	−	4
=	○		20

As for 13,463,512×1.8=24,234,321.6:
Direction: Attach 8 from the bottom
(13,463,512+13,463,512×0.8)

As for
34,645,734×0.9=3,118,116.6—
Another direction: Subtract 1 from the bottom and you will obtain the same result.
3,464,573.4−3,464,573.4×0.1)

As for
43,675.2÷1.2=36,396—
Direction: First put the original number, halve and divide it by 6.

As for
43,675.2÷1.2=36,396—
Another direction: Subtract 2 from outside itself.
(43,675.2−36,396×0.2)

As for
3,463,335×35=121,216,725—
Direction: Multiply it by 5 and 7.
(3,463,335×5×7)
Another direction: Multiply it by 7 and then halve it, and you will get the same result.
Thus (3,463,335×70÷2)

As for
3,463.5×42=145,467—
Direction: Multiply it by 6 and 7.
Thus (3,463.5×6×7)

As for
13,465.43×135=1,817,833.05—
Direction: Multiply it by 3 and 9, and by halving it, you will get the result.
Thus (13,465.43×30×9÷2)
Another direction: Multiply it by 9 and then by 5, and you will get the same result.
Thus (13,465.43×9×15)

As for

56,473.5×5—

Direction : Multiply it by 5 from the bottom, and you will get the result.

This method shows a marked change in the calculative method, against putting the product or quotient in the position of the dividend, over against the previous method, as shown in Section 1, of putting the dividend, divisor, and quotient in three positions.

This method is what is called Chung-yin 重因 (double division) in the *Yang-hui-suan-fa* 楊輝算法, 1274, which tries to analyse a divisor of many places into those of a single place and thereby to simplify the calculative method.

From the above exercises, it is evident that in multiplication of a single order divisor, multiplication is operated beginning with the top (number) of the dividend. But this is not the only set rule.

This shows that, by this time, in multiplication of one place divisor, the ancient three-row calculative method has been replaced by that of operating by turning the dividend into the product.

The following passage occurs in the Suan-fa-t'ung-pien-pên-mo 算法通變本末 Vol. 1, 1274, by YANG Hui 楊輝

單因，法曰，置衆位爲實，陰記單位爲法，從上位因起，言十過身，言如就身改之

" When the multiplier is of a single place, the method says : Put down the multiplicand as the quotient, memorize the multiplier, start multiplying from the top ; when *shih* 十 (tens) is called, put it above itself and when *ju* 如 (itself) is called, change it in itself." In case the multiplier is of a single place, do not put it down, but commit it to memory, and the multiplicand is turned into the product. Probably this is a method also adopted by HSIA-HOU Yang 夏侯陽.

Now comes division of a single place number. In ancient mathematical works such as the *Chiu-chang-suan-shu* 九章算術, *Sun-tzŭ-suan-ching* 孫子算經 and *Wu-ts'ao-suan-ching* 五曹算經 is found the phrase " pan-chih " 半之 (to halve this). calculative method of halving a number probably implied halving off-hand a number put down.

Thus, in division, in the case of a single-place divisor, calculative simplification was practised only by halving.

However, the introduction of a calculative simplification in case of a single-place multiplier inevitably gave rise to a calculative simplification in division also.

As to culculative simplification in division, the *Mêng-ch'i-p'i-tan* 夢溪筆談 by CH'EN Kua 沈括 (1030–1094) of the North-Sung period, says :

算術多門，如求一，上驅，塔因，重因之類皆不離乘，除，惟增成一法稍異其術，都不用乘，除，但補虧就盈而已，假如欲九除者增一便是，八除者增二便是

" Arithmetic is divided into a number of branches. Ch'iu-i 求一, Shang-chü,

上驅, T'a-yin 塔因, and Chung-yin 重因 are all branches of multiplication and division. In my view, however, the Tsêng-ch'êng-i-fa 增成一法 is somewhat different; it has nothing to do with multiplication or division. It deals with the shortage of a number and supplements it. For example, to divide by 9, 1 is added; to divide 8, 2 is added."

For example:

10÷8	
10	1×2 is added
+2	
12	
+4	2×2 is added
124	
+8	2×4 is added
1248	
−8	8 is subtracted
+10	10 is added
125	

Then it says: 算術不患多學, 見簡卽用, 見繁卽變, 不膠一法乃爲通術也 "One must not grudge taking the trouble of learning mathematics. When a method is simple enough, adopt it. When a method is too complicated, simplify it. If one is not glued to one particular method, one may be versed in it."

This shows that it developed from the Tsêng-ch'êng-i-fa 增成一法 Kuei-i-ch'u-fa 歸一除法 of later days to the Chiu-kuei-fa 九歸法 (Division by one-figure numbers accompanied by division-rhymes).

3. Advent of the Chiu-kuei-fa 九歸法

The origin of the Chiu-kuei 九歸 method is not definitely known, no accurate source being accessible. However, the Suan-fa-t'ung-pien-pên-mo 算法通變本末, the First Volume, (1274) by Yang Hui 楊輝 says:

諸家算書, 用度不出乘・除・開法三法, 起例不出如, 十二字, 下算不出橫・直二位引而伸之, 其機殆無窮盡矣, 乘除者鈎深致遠之法, 指南算法以加・減・九歸・求一, 旁求捷徑, 學者豈容不兼而用之

"According to the mathematical books by various writers, calculation is limited to the following three methods: multiplication, division, and extraction of roots. Chi-li 起例 is limited to the use of the two letters *ju* 如 and *shih* 十. Block-arrangement is limited to horizontality and verticality. If extended, it would know almost no limit in inventions. Multiplication and division, originally, aimed at reaching for the depth and thorough study of the distant. The *Chih-nan-suan-fa* 指南算法 describes addition, subtraction, chiu-kuei 九歸 and ch'iu-i 求一 and also aims at mastering methods simpler than these. How can a student neglect combined mastery of all these methods?"

Though the mathematical works by various authors refer only to the method of Ch'êng-ch'u-k'ai-fa 乘除開法 (In this case ch'u 除 refers to Shang-ch'u 商除), the book points out that the *Chih-nan-suan-fa* 指南算法 alone describes the methods of chia-chien 加減, chiu-kuei 九歸, ch'iu-i 求一 and point out that there was a method of simplification. In this case, chia 加 (addition) and chien 減 (subtraction) refer to Shên-wai-chia-fa 身外加法 (addition from outside itself), and Shên-wai-chien-fa 身外減法 (subtraction from outside itself).

According to YANG Hui 楊輝, therefore, the oldest mathematical work that records the chiu-kuei 九歸 method is the *Chih-nan-suan-fa* 指南算法. Though this book has been lost sight of since, an account which would enable us to infer the date of the publication of this book is found under the item *Suan-hsüeh-yüan-liu* 算學源流 as the end of the *Suan-fa-t'ung-tsung* 算法統宗, (Preface, 1592.) From this it may be known that the book was published in 1078–1189. It reads :

元豐紹興淳熙以來，刊刻者多且見聞者著之 . . . . . 議古根源，益古算法，證古算法，明古算法，辨古算法，明源算法，金科算法，指南算法，應用算法，曹唐算法，賈憲九章，通徵集，通機集，盤珠集，走盤集，三元化零歌，鈐經，鈐釋

" Since Yüan-fêng 元豐, Shao-hsing 紹興, and Chun-hsi 淳熙 a number of men have written books, and those who have read or heard of them have written also. The list reads : *I-ku-kên-yüan* 議古根源, *I-ku-suan-fa* 益古算法, *Chêng-ku-suan-fa* 證古算法, *Ming-ku-suan-fa* 明古算法, *Pien-ku-suan-fa* 辨古算法, *Ming-yüan-suan-fa*, 明源算法, *Chin-k'o-suan-fa* 金科算法, *Chih-nan-suan-fa* 指南算法, *Ying-yung-suan-fa* 應用算法, *Ts'ao-t'ang-suan-fa* 曹唐算法, *Chia-hsien-chiu-chang* 賈憲九章, *T'ung-wei-chi* 通徵集, *T'ung-chi-chi* 通機集, *P'an-chu-chi* 盤珠集, *Tsou-pan-chi* 走盤集, *San-yüan-hua-ling-ko* 三元化零歌, *Ling-ching* 鈐經, *Ling-shih* 鈐釋."

I have said that in multiplication, with a single place multiplies a calculative simplification was invented and that in division, also with a single place divisor. A calculative simplification inevitably came. This will be investigated here more carefully.

YANG Hui 楊輝 in his Chiu-kuei-hsiang-shuo 九歸詳說 (Detailed Discussion of the Chiu-kuei Method of *Ch'êng-chu-t'ung-pien-suan-pao* 乘除通變算寶), the Middle Volume, (1274), says :

一位爲法爲除，則用九歸代之，若兩三位商除自合，伸引歸法取用，今人以第一位用歸，以第二位第三位仍用商除，是一題涉二法也

" In division, when a single place divisor, the Chiu-kuei 九歸 method may replace the Shang-ch'u 商除 method, when a two or more place divisor, the Shang-chŭ method will serve. Nowadays, with the first place, they use the Chiu-kuei 九歸 method, but with the second and third they use the Shang-ch'u 商除 method. This means two methods are applied on one problem."

The Chiu-kuei 九歸 method was invented as simplified calculative method for division with a single place divisor; prior to this, in case of a divisor of two or more places, the Shang-ch'u 商除 method, the calculative method given in the *Sun-tzŭ-suan-ching* 孫子算經 was used; but around YANG Hui's time in the last days of the Sung dynasty, the Chiu-kuei method was now extended to a divisor of two or more places. This point is definitely proved.

Even after the extensive use of the Chiu-kuei method to a divisor of more than two places, until the advent of the phrase Chung-kuei-huan-yüan 撞歸還原 in the last part of the Sung period, the value of the Chiu-kuei method was that of a mere simplifying method. The Shang-ch'u 商除 method was the normal method of division.

YUAN Hui's *Suan-fa* 楊輝算法 discusses in the Suan-fa-t'ung-pien-pên-mo 算法通變本末, First Volume, Hsiang-ch'êng 相乘 and Shang-ch'u 商除 in *Ch'êng-chu-t'ung-pien-suan-pao* 乘除通變算寶, Middle Volume, Chia-fa 加法, Chien-fa 減法, Chiu-i-tai-chêng-ch'u 求一代乘除, and Chiu-kuei 九歸 and Suan-wu-ting-fa 算無定法, in Fa-suan-ch'ü-yung-pên-mo 法算取用本末 the Last Volume, ...Application of the simplifying method.

In this case, chia-fa 加法 (addition) and chien-fa 減法 (subtraction) referred to Shên-wai-chia-fa 身外加法 (addition to outside itself) and Shên-wai-chien-fa 身外減法 (subtractive on from outside itself) which correspond to our present Shêng-i-chêng-fa 省一乘法 and Shêng-i-chu-fa 省一除法 while the Chiu-kuei was learned after the Shang-ch'u 商除 or the Chien-fa (method of division).

Again, in the *Fa-suan-ch'ü-yung-pên-mo* 法算取用本末, Last Volume, this passage occurs:

夫算者題從法取法將題問，凡欲見明一法必設一題，若遇問題須詳取用，大概不出乘除，後人用加・減・歸・折，乃乘除之曲徑也

"One who calculates should follow the methods, and adoption depends upon the kind of a question. Should one desire to be versed in a method, he should devise a question and conduct exhaustive calculations. They are mostly within the spheres of multiplication and division. Modern people resort to chia 加 (addition), chien 減 (subtraction), kuei 歸 (chiu-kuei division), chê 折 (halving), which are detours of calculations."

In the Ting-i-hsiang-shuo 定位詳說, *Chêng-chu-t'ung-pien-suan-pao* 乘除通變算寶, Middle Volume, a passage reads:

視題用法本無定，據所用因・折・加・減・歸・損各有定位，若諸法互用重用，定位殆將不可律論矣，恐爲學者惑，今立定率術日，先以乘除本法，定所得位，訖而後重互雜法，必無誤礙也

"There is no set rule to use a certain method in handing a problem. For each yin 因 (multiplication), chê (halving), chia 加 (addition), chien 減 (subtraction), kuei 歸 (division), and sun 損 (loss) used, a proper position is fixed. For if several methods used similar positions mutually and arbitrarily, there would be no ground for argumentation, and this would confuse the student." The Ting-shuo-shu 定率術 is now established. If you first establish the position which has been properly obtained by the regular methods of multiplication and division. This refers to Hsiang-chêng 相乘 and shang-ch'u 商除, and then use the minor methods, there would arise no confusion."

The Chiu-kuei method, in the former, is called a heretical method, while in the latter, the Shang-ch'u method is treated as the orthodox method and the Chiu-kuei a minor method.

Even the *Suan-hsüeh-ch'i-mêng* 算學啓蒙 (1299) which mentions the chiu-kuei method alone as the only method of division says :—

按古法多用商除，爲初學者難入，則後人以此法代之，卽非正術也

" A study of ancient books shows that most of them adopted the Shang-ch'u method, but as it was difficult for beginners, this method was adopted as a substitute for it. Therefore, it is not an orthodox method."

The situation is clear from this quotation.

4. The Chiu-kuei-ko-chüeh 九歸歌訣 (division by the number of a figure accompanied by division rhymes) at the earliest stage

The earliest extant book that records the Chiu-kuei-ko-chüeh is *Jih-yung-suan-fa* 日用算法 (1262). This book is now lost sight of, but some exercises are included in the *Chu-chia-suan-fa* 諸家算法.

The Pa-kuei 八歸 (Division by 8) in division using division rhymes.

As for $6800 \div 16 = 425$—

Explanation : There are five ways.

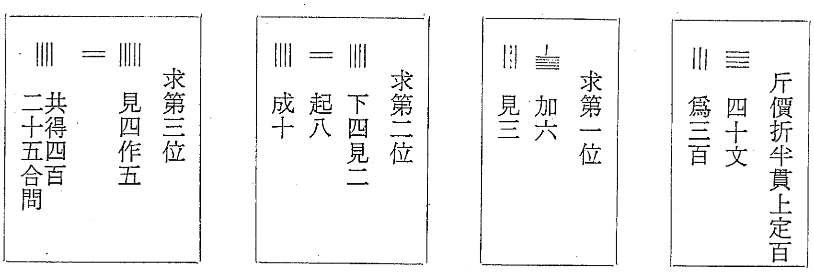
A third direction : Halve and divide by 8.

Five solutions are offered. As to the third, the following is given :

三日，斤價爲實折半，取十六兩爲八兩價，八歸是取八兩爲一兩價

草 見 後 圖

Diagram for quick reckoning



$6800 \div 16 = \left(\frac{6800}{2}\right) \div \left(\frac{16}{2}\right)$; therefore, $3400 \div 8$

(1) Chien-san-hsia-liu 見三下六 (See-three-add-six-down)	34 + 6 3^{5}_{5}
(2) Chi-pa-chêng-shih 起八成十 (Remove-eight-make-ten-one-up) Chien-êrh-hsia-ssŭ 見二下四 (See-two-add-four-down)	− 8 +10 42 + 4 424
(3) Chien-ssŭ-tso-wu 見四作五 (See-four-make-five)	− 4 + 5
(4)	425

In this case, the Shang-ch'u 商除 method of the divisor 16 is not used, but the Chiu-kuei-ko-chüeh 九歸歌訣 is used, (a half of 16 is used).

Yang Hui 楊輝 in his *Ch'êng-ch'u'u-t'uug-pien-suan-pao* 乘除通變算寶, Middle Volume, says :—

今人以第一位用歸，以第二位第三位仍用商除

"Nowadays, with the first place, they use the Kuei 歸 method, but with the second and third they use the Shang-ch'u 商除 method."

So it seems that, in division with a divisor of two or more places, the Chiu-kuei-ko-chüeh method was not em-ployed. Probably because of the absence of the rhymes for the Ch'ung-kuei 憧歸 method, the Huan-yüan 還元 method, division by the Chiu-kuei-ko-chüeh alone would have been too difficult. This method will be described in the following.

The oldest full record of the Chiu-kuei-ko-chüeh 九歸歌訣 is the *Ch'êng-ch'u-t'ung-pien-suan-pao* 乘除通變算寶, Middle Volume, (1274) by Yang Hui 楊輝. A passage in it says :

九歸新括，以古句今注，兩存之

"The Chiu-kuei rhymes are ancient rhymes, newly annotated. Both forms are preserved."

Kuei-shu-ch'iu-ch'êng-shih 歸數求成十 (Divide-number-make-ten),

Chiu-kuei 九歸—(division by 9) Yü-chiu-ch'êng-shih 遇九成十 (Meet-nine-make-ten), (Note, 9÷9, →

遇九	− 9
成十	+10
	1

Pa-kuei 八歸—(division by 8) Yü-pa-ching-shih 遇八成十 (Meet-eight-make-ten),

Ch'i-kuei 七歸—(division by 7) Yü-ch'i-ch'êng-shih 遇七成十 (Meet-seven-make-ten),

Liu-kuei 六歸—(division by 6) Yü-liu-ch'êng-shih 遇六成十 (Meet-six-make-ten),

Wu-kuei 五歸—(division by 5) Wu-kuei-ch'êng-shih 遇五成十 (Meet-five-make-ten),

Ssŭ-kuei 四歸—(division by 4) Yü-ssŭ-ch'êng-shih 遇四成十 (Meet-four-make-ten), San-kuei 三歸—(division by 3) Yü-san-ch'êng-shih 遇三成十 (Meet-three-make-ten),

Êrh-kuei 二歸—(division by 2) Yü-êrh-ch'êng-shih 遇二成十 (Meet-two-make-ten)

Kuei-ch'u-tzŭ-shang-chia 歸除自上加 (Division-to-adding-itself)

Chiu-kuei 九歸—(division by 9) Chien-i-hsia-i 見一下一 (See-one-add-one-down),

Chien-êrh-hsia-êrh 見二下二 (See-two-add-two-down),

Chien-san-hsia-san 見三下三 (See-three-add-three-down),

Chien-ssŭ-hsia-ssŭ 見四下四 (See-four-add-four-down).

Pa-kuei 八歸—(division by 8) Chien-i-hsia-êrh 見一下二 (See-one-add-two-down)

Chien-êrh-hsia-ssu 見二下四 (See-two-add-four-down)

Chien-san-hsia-liu 見三下六 (See-three-add-six-down)

Ch'i-kuei 七歸 (division by 7) Chien-i-hsia-san 見一下三 (See-one-add-three-down)

Chien-êrh-hsia-liu 見二下六 (See-two-add-six-down),

Chien-san-hsia-êrh-chi-chiu 見三下十二即九 (See-three-below-twelve, namely-add-nine-down)

(Note: 30÷7, (1)

	3
見三下十二...	12
	42

(2)

	3	
見三下九...	9	...See-three-add-nine-down
	39	
遇七成十...	−7	...Meet-Seven-make-ten
...	10	
	42	

4 is quotient and 2 is remainder.)

Liu-kuei 六歸—(division by 6) Chien-i-hsia-ssŭ 見一下四 (See-one-add-four-down),

Chien-êrh-hsia-shih-êrh-chi-pa 見二下十二即八 (See-two-below-twelve, namely-add-eight-down)

Wu-kuei 五歸—(division by 5) Chien-i-tso-êrh 見一作二 (See-one-make-two),

Chien-êrh-tso-ssŭ 見二作四 (See-two-make-four)

Ssŭ-kuei 四歸—(division by 4) Chien-i-hsia-shih-êrh-chi-liu 見一下十二即六 (See-one-below-twelve, namely-add-six-down)

San-kuei 三歸—(division by 3) Chien-i-hsia-shih-êrh-chi-ch'i 見一下十二即七 (See-one-below-twelve, namely-add-seven-down)

Pan-êrh-wei-wu-chi 半而爲五計 (Halve and-make-five)

Chiu-kuei 九歸—(division by 9) Chien-ssŭ-wu-tso-wu 見四五作五 (See-forty-five-make-five) →

Pa-kuei 八歸—(division by 8) Chien-ssŭ-tso-wu 見四

(Note, 45÷9,

	45
見四五...	−45
作五 ...	+5
	5

5 is quotient.)

作五 (See-four-make-five)

Ch'i-kuei 七歸—(division by 7) Chien-san-wu-tso-wu 見三五作五 (See-thirty-five-make-five)

Liu-kuei 六歸—(division by 6) Chien-san-tso-wu 見三作五 (See-three-make-five)

Wu-kuei 五歸—(division by 5) Chien-êrh-wu-tso-wu 見二五作五 (See-twenty-five-make-five)

Ssŭ-kuei 四歸—(division by 4) Chien-êrh-tso-wu 見二作五 (See-two-make-five)

San-kuei 三歸—(division by 3) Chien-i-wu-tso-wu 見一五作五 (See-fifteen-make-five)

Êrh-kuei 二歸—(division by 2) Chien-i-tso-wu 見一作五 (See-one-make-five)

Ting-i-t'ui-wu-ch'a 定位退無差 (Fix-place-recede-no-difference)

The book does not describe the calculative method by means of the Chiu-kuei-ko-chüeh 九歸歌訣. In order to compute 648÷8 = 81, the following is necessary.

					Note
(1)	The figures are put down :—	(1) (2)	𝍥 𝍬 𝍧	(1) (2)	648 −40 +5
(2)	Chien-ssŭ-tso-wu 見四作五 (See-four-make-five)	(3)	$\overset{五}{二}$ 𝍬 𝍧	(3)	$^{5}_{2}48$ 4
(3)	Chien-êrh-hsia-ssŭ 見二下四 (See-two-add-four-down)	(4)	$\overset{五}{二}$ 𝍰 𝍧	(4)	$^{5}_{2}88$ − 8 +10
(4)	Yü-pa-ch'êng-shih 遇八成十 (Meet-eight-make-ten)	(5)	八 〇 𝍧	(5)	808 − 8 − 10
(5)	Yü-pa-ch'êng-shih 遇八成十 (Meet-eight-make-ten)		八 壹		81

It would have been impossible to do this on an abacus.

Let us investigate the extent to which the Chiu-kuei-ko-chüeh 九歸歌訣 was used in these days.

Out of the 300 questions in the *Fa-suan-ch'ü-yung-pên-mo* 法算取用本末, Last Volume, two may be cited.

As for 4968÷54 = 92—

Direction : With 54 as divisor, let liu-kuei 六歸 (division by 6) and chiu-kuei 九歸 (division by 9) serve the purpose. If the calculation by liu-kuei is troublesome, you may add 5 on the place and multiply it twice by means of chiu-kuei 九歸 (division by 9).

As for 49152÷192 = 256—

Direction : Generally, chien-êrh 減二 (subtracting 2) and chien-liu 減六 (subtracting 6) are used; when chien-lin is used, what is halved may be divided by pa-kuei 八歸 ; and if pa-kuei 八歸 is found troublesome, it may be replaced by chia-êrh-wu 加二五 (add-twenty-five), or by halving three times and use of chia-êrh-wu 加二五.

As for the former question, namely, 4968÷54 is 4968÷6÷9, it says "If liu-kuei 六歸 (division by 6) is found troublesome, try 4968×15÷9÷9 ", and as for the latter 49152÷192, it says : Try 49152÷12÷16, and if division by 2 is troublesome, halve and then divide it by 6,—namely, 49152÷2÷6÷16 ; and if division by 6 is troublesome, multiply it by 15 and divide it by 9,—namely, 49152÷2×15÷9÷16 ; if division by 16 is troublesome, halve and divide it by 8—(a÷16=a÷2÷8) ; and if division by 8 is troublesome, multify it by 125 (a÷8 : a×125) ; Otherwise, halve it 3 times instead of multiplying it 125 times —(a÷2÷2÷2) " and so forth. Thus it is shown that the method by Chiu-kuei-ko-chüeh 九歸歌訣 (the chiu-kuei division-rhymes) was by no means a simple way of calculation.

(Note. The passage says " Multiply it by 15 and divide it by 9 instead of dividing it by 6 ; and multiply it by 125 instead of dividing it by 8." Nowadays, it would be much simpler to divide by 6 or 8. However, at that time there already existed the method later called Shên-wai-chia-fa 身外加法 (Addition to outside itself) therefore, in multiplying by 15 was operated by omitting the first dividor number and multiplying by 5, and in multipling by 125, this was operated by omitting the first dividor number and multiplying by 25. For this reason, it was a simpler method.)

As previously stated, YANG Hui 楊輝, remarking. "九歸新括, 以古句今注, 兩存之 " (The Chiu-kuei-hsin-kua are ancient rhymes, newly annotated. Both forms are preserved.), introduces 32 chiu-kuei 九歸 rhymes. This shows that, over against YANG Hui's new rhymes, ancient rhymes had existed prior to YANG Hui.

Now, let us investigate what kind of ancient rhymes had existed. YANG Hui himself, in the *Suan-fa-t'ung-pien-pên-mo* 算法通變本末, First Volume, says :

學九歸, 若記四十四句, 念法非五七日不熟, 今但於詳解算法九歸題術中, 細看注文, 便知用意之隙, 而念法用法一日可記矣

" To master the Chiu-kuei 九歸 method, if you memorized 44 rhymes to learn the method, it used to take 35 days. Nowadays, however, if you study carefully the notes on the *Chiu-kuei-t'i-shu* 九歸題術 in the *Hsiang-chieh-suan-fa* 詳解算法, if you know how to pay attention, one day would be enough to memorize the rules of operating this method."

So it would seem that thea ncient rhymes which had existed prior to the new 32 rhymes numbered 44.

A rhyme, which might be regarded as one of the ancient division-rhymes, not included in the Chiu-kuei-hsin-kua, occurs in one of the exercises represented in the *Fa-suan-ch'ü-yung-pên-mo* 法算取用本末, Last Volume.

As for $675,700 \div 2330 = 290$—Direction says: Chien-êrh-hsia-liu-chia-huan-êrh 見二下六加還二 (See-two-add-six-down-return-two); and Chien-liu-hsia-êrh-shih-ssŭ-chia-huan-pa-po 見六下二十四加還八百 (See-six-below-twenty-four, return 800.)

$$
\begin{aligned}
6757 \div 233 &= (6757 \times 3) \div (233 \times 3) \\
&= 20271 \div 699 \\
&= 20271 \div (700-1)
\end{aligned}
$$

In this way, here is applied a kind of calculation we term kuei-i-hun-yung-ch'u-fa 歸一混用除法. Now, in this rhyme for Ch'i-kuei 七歸 is used a rhyme Chien-liu-hsia-êrh-shih-ssŭ 見六下二十四 (See-six-below-twenty-four). This is not included in the above-mentioned Chiu-kuei-hsin-kua (New chiu-kuei rhymes). From this example, the following 44 rhymes may be inferred as those ancient division-rhymes which had existed prior to Yang Hui's time.

Chiu-kuei 九歸 (division by 9):

Chien-i-hsia-wu 見一下一 (See-one-add-one-down)
Chien-êrh-hsia-êrh 見二下二 (See-two-add-two-down)
Chien-san-hsia-san 見三下三 (See-three-add-three-down)
Chien-ssŭ-hsia-ssŭ 見四下四 (See-four-add-four-down)
Chien-wu-hsia-wu **見五下五** (See-five-add-five-down)
Chien-liu-hsia-liu **見六下六** (See-six-add-six-down)
Chien-ch'i-hsia-ch'i **見七下七** (See-seven-add-seven-down)
Chien-pa-hsia-pa **見八下八** (See-eight-add-eight-down)
Yü-chiu-ch'êng-shih 遇九成十 (Meet-nine-make-ten)

Pa-kuei 八歸 (division by 8)

Chien-i-hsia-êrh 見一下二 (See-one-add-two-down)
Chien-êrh-hsia-ssŭ 見二下四 (See-two-add-four-down)
Chien-san-hsia-liu 見三下六 (See-three-add-six-down)
Chien-ssŭ-tso-wu 見四作五 (See-four-make-five)
Chien-wu-hsia-shih-êrh **見五下十二** (See-five-below-twelve)
Chien-liu-hsia-shih-ssŭ **見六下十四** (See-six-below-fourteen)
Chien-chi-hsia-shih-liu **見七下十六** (See-seven-below-sixteen)
Yü-pa-ch'êng-shih 遇八成十 (Meet-eight-make-ten)

Ch'i-kuei 七歸 (division by 7)

Chien-i-hsia-san 見一下三 (See-one-add-three-down)
Chien-êrh-hsia-liu 見二下六 (See-two-add-six-down)
Chien-san-hsia-shih-êrh 見三下十二 (See-three-below-twelve)
Chien-ssŭ-hsia-shih-wu **見四下十五** (See-four-below-fifteen)

Chien-wu-hsia-êrh-shih-i **見五下二十一** (See-five-below-twenty-one)

Chien-liu-hsia-erh-shih-ssŭ **見六下二十四** (See-six-below-twenty-four)

Yü-ch'i-ch'êng-shih 遇七成十 (Meet-seven-make-ten)

Liu-kuei 六歸 (division by 6)

Chien-i-hsia-ssŭ 見一下四 (See-one-add-four-down)

Chien-êrh-hsia-shih-êrh 見二下十二 (See-two-below-twelve)

Chien-san-tso-wu 見三作五 (See-three-make-five)

Chien-ssŭ-hsia-êrh-shih-ssŭ **見四下二十四** (See-four-below-twenty-four)

Chien-wu-hsia-san-shih-êrh **見五下三十二** (See-five-below-thirty-two)

Yü-liu-chêng-shih 遇六成十 (Meet-six-make-ten)

Kuei-wu 五歸 (division by 5)

Chien-i-tso-êrh 見一作二 (See-one-make-two)

Chien-êrh-tso-ssŭ 見二作四 (See-two-make-four)

Chien-san-tso-liu **見三作六** (See-three-make-six)

Chien-ssŭ-tso-pa **見四作八** (See-four-make-eight)

Ssŭ-kuei 四歸 (division by 4)

Chien-i-hsia-shih-êrh 見一下十二 (See-one-below-twelve)

Chien-êrh-tso-wu 見二作五 (See-two-make-five)

Chien-san-hsia-ssŭ-shih-êrh **見三下四十二** (See-three-below-forty-two)

Yü-ssŭ-tso-pa 遇四作八 (Meet-four-make-eight)

San-kuei 三歸 (division by 3)

Chien-i-hsia-êrh-shih-i 見一下二十一 (See-one-below-twenty-one)

Chien-êrh-hsia-ssŭ-shih-êrh **見二下四十二** (See-two-below-forty-two)

Yü-san-ch'êng-shih 遇三成十 (Meet-three-make-ten)

Êrh-kuei 二歸 (division by 2)

Chien-i-tso-wu 見一作五 (See-one-make-five)

Yü-êrh-tso-shih 遇二成十 (Meet-two-make-ten)

Note 1. It is possible that Yü-chiu-ch'êng-shih 遇九成十 (Meet-nine-make-ten), Yü-pa-chêng-shih 遇八成十 (Meet-eight-make-ten),......Yü-êrh-ch'êng-shih 遇二成十 (Meet-two-make-ten) were respectively recited Chi-chiu-chêng-shih 起九成十 (Remove-nine-make-ten-one-up), Chi-pa-chêng-shih 起八成十 (Remove-eight-make-ten-one-up), and Chi-êrh-chêng-shih...... 起二成十 (Remove-two-make-ten-one-up). See the *Jih-yung-suan-fa* 日用算法

Note 2. Heavier type indicates those not included in YANG Hui's New Rhymes.

Note 3. Those appearing in YANG Hui's Hsin-kua 新括, but not included in this list, are as follows :—

Chiu-kuei 九歸 (division by 9)

Chien-ssŭ-shih-wu-tso-wu 見四五作五 (See-forty-five-make-five)

Ch'i-kuei 七歸 (division by 7)

Chien-san-shih-wu-tso-wu 見三五作五 (See-thirty-five-make-five)

Wu-kuei 五歸 (division by 5)

Chien-êrh-shih-wu-tso-wu 見二五作五 (See-twenty-five-make-five)

San-kuei 三歸 (division by 3)

Chien-i-shih-wu-tso-wu 見一五作五 (See-fifteen-make-five)

Note 4. The difference from Yang Hui's New Rhymes may be seen in the use of the word chi 卽 (Namely) in the following rhymes :—

Ch'i-kuei 七歸 (division by 7)

Chien-san-hsia-shih-êrh-chi-chiu 見三下十二卽九 (See-three-below-twelve, namely-add-nine-down)

Liu-kuei 六歸 (division by 6)

Chien(êrh-hsia-êrh-chi-pa 見二下十二卽八 (See-two-below-twelve, namely add-eight-down)

Ssŭ-kuei 四歸 (division by 4)

Chien-i-hsia-shih-êrh-chi-liu 見一下十二卽六 (See-one-below-twelve, namely-add-six-down)

San-kuei 三歸 (division by 3)

Chien-i-hsia-êrh-shih-i-chi-ch'i 見一下二十一卽七 (See-one-below-twenty-one, namely-add-seven-down)

It may be supposed that Yang Hui, considering the 44 ancient rhymes too difficult for memorizing, introduced the idea of supplementary numbers into the Chiu-kuei-ko-chüeh 九歸歌訣, and therefore coined new rhymes. For, in his explanation of Sun-ch'êng 損乘 method, he says this :—To multipy a number by 3 will be equal to subtracting from the multiplicant $\times 10$ the multiplicand $\times 7$.

For example : 2×3—

$$\begin{array}{r} 20 \\ -14 \\ \hline 6 \end{array}$$ ....(2×7) is subtracted

In these cases all decimal fractions are ignored. Therefore, a more precise statement would be " to multiply the multiplicand by 10 and subtract from the bottom place the multiplier $\times$ the supplementary number.) He says :

. . . 三乘者損七，二乘者損八，竝自末位求起，卽下乘也，是反用九歸之術

" Multiplication by 3 loses 7 ; and multiplication by 2 loses 8. From the lowest place operation is started ; it is hsia-ch'êng 下乘. This is a reverse operation of the Chiu-kuei 九歸 method." He also says :

學九歸，若記四十四句，念法非五七日不熟，今但於詳解算法九歸題術中，細看注文，便知用意之隙，而念法用法一日可記矣

" To master the Chiu-kuei 九歸 method, if you memorized 44 rhymes to learn the method, it used to take 5×7 (35) days. Nowadays, however, if you study carefully the Notes on the Chiu-kuei-t'i-shu 九歸題術 in the *Hsiang-chieh-suan-fa* 詳解

算法, and you know how to pay attention, one day would be enough for memorizing the rules of operating this method."

When YANG says: "The Chiu-kuei rhymes are ancient rhymes, newly annotated. Both forms are preserved," he is referring to such a fact as that; among the rhymes Ch'i-kuei 七歸, and Chien-san-hsia-shih-êrh-chi-chiu 見三下十二卽九 (See-three-below-twelve, namely-add-nine-down), Chien-san-hsia-shih-êrh 見三下十二 (See-three-below-twelve) is an old rhyme, while Chien-san-hsia-chiu 見三下九 (See-three-add-nine-down) is a new one.

Such a rhyme from the Chiu-kuei-hsin-kua 九歸新括 (The New Chiu-kuei Rhymes) is not found in any later mathematical book where the old rhyme Ch'i-kuei 七歸: Chien-san-hsia-shih-êrh 見三下十二 (See-three-below-twelve) is found as Ch'i-san-ssŭ-shih-êrh 七三四十二 (Seven-three-make-forty-two). Chien-san-hsia-chiu 見三下九 (See-three-add-nine-down) is 3÷7, namely three-nine 39, also Yü-ch'i-ch'êng-shih 遇七成十, namely, make four-two 42 (4 is quotient and 2 is remainder) (Note. The sixteen rhymes for 歸除自上加 (Division by adding to itself) given by YANG Hui may be so readily memorized. They will release you from the trouble of memorizing Ch'i-kuei 七歸 (division by 7); Chien-san-hsia-shih-êrh 見三下十二 (See-three-below-twelve), Liu-kuei 六歸 (division by 6): Chien-êrh-hsia-shih-êrh 見二下十二, Ssŭ-kuei 四歸 (division by 4): Chien-i-hsia-shih-êrh 見一下十二 (See-one-below-twelve), and San-kuei 三歸 (division by three): Chien-i-hsia-êrh-shih-i 見一下二十一 (See-one-below-twenty-one), for all you have to do is to add supplementary numbers below,—that is, in Ch'i-kuei 七歸 (division by 7), the multiplication rhyme for 3 the supplementary number of 7 and 3 in Chien-san 見三 (See-three) being 9, you simply add 9 below, the multiplication rhyme for 4 the supplementary number of 6 and 2 in Chien-êrh 見二 (See-two) being 8, you simply add 8 below; the supplementary number of 4 is 6, and i 一 (one) in Chien-i 見一 (See-one), namely 6; the supplementary number of 3 is 7, and i 一 (one) in Chien-i 見一 (See-one) namely 7. While it took you 35 days to memorize old reckoning rhymes, no wonder it would take you only one day to do the new method.)

As previously stated, prior to YANG Hui's time there existed no reckoning rhymes for Ch'ung-kuei-huan-yüan 撞歸還原.

5. Division without the Ch'ung-kuei-huan-yüan rhymes

As previously mentioned, the earliest Chiu-kuei-ko-chüeh 九歸歌訣 (Division rhymes) did not include any Ch'ung-kuei-huan-yüan rhyme. It is true, YANG Hui 楊輝 says:

今人以第一位用歸，以第二位第三位仍用商除，是一題涉二法也

"Nowadays, with the first place, they use the Chiu-kuei 九歸 method, but with the second and third they use the Shang-ch'êng 商除 method. This means that two methods are applied on one problem."

The *Hsiang-ming-suan-fa* 詳明算法 (1373) published a century after YANG Hui 楊輝 jointly by AN Chi-ch'i 安止齋 and Ho Ping-tzŭ 何平子 describes this method. It will be introduced in the following.

The book prints as k'ou-shou 口授 (dictated) the following passage :—

撞歸起一不同，以五歸言之，撞歸見五即五歸歸不得無除作九是也，五歸見四無除，則五四倍作八，却起一於下位還五，以除如數少，又起一還五以除也

" The Ch'ung-kuei 撞歸 method is not similar to the Chi-i 起一. Speaking from the standpoint of the Wu-kuei 五歸, the Ch'ung-kuei 撞歸 is this : Seeing 5 and dividing it by 5 and there is no number for division left ; Wu-ch'u-tso 無除作九 (If no dividend, make 9) refers to this. If seeing 5 and dividing it by 5 Wu-ssŭ-pei-tso-pa 五四倍作八 (Five-four-make-eight) refers to this and there is no number for division left, subtract one and return five to the lower place. In dividing, if the dividend is too small a number, raise one again, and return five ; by which the number is divided."

Toward the end of the explanation of the Kuei-ch'u (division method), it says :

唯，或歸而無除，則起此已歸一算，而以元數復置於下位，然後視其多寡而算之，又撞歸之法皆變通之術也，亦不可不知，今具列如後，見二無除作九二，見三無除作九三，見四無除作九四，見五無除作九五，見六無除作九六，見七無除作九七，見八無除作九八，見九無除作九九

" However, when there is no dividend, subtract the amount already calculated, put the original number on the lower place. Then study the amount and calculate. By the way, the Ch'ung-kuei 撞歸 method is only one of the varied methods. It is indispensable. The details are listed below.

Chien-êrh-wu-ch'u-tso-chiu-êrh 見二無除作九二 (See-two-no-dividend-make-nine-add-two-down)...... Chien-chiu-wu-ch'u-tso-chiu-chiu 見九無除作九九 (See-nine-no-dividend-make-nine-add-nine-down)."

Only the *Preface* says :

夫學者初習因歸，則口授心會，至於撞歸起一，時有差謬

" As the student first learns multiplication and division (of the single place), he should learn by reciting. When he comes to the Ch'ung-kuei 撞歸 and Chi-i 起一 methods, he would sometimes make errors."

For this reason, explanations of multiplication and division are made without the Ch'ung-kuei-huan-yüan 撞歸還原 method.

As for $6589 \div 55 = 1198$—

丅 ||||| 丌 ≐ (6 5 8 9)	(1)	Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten) I-wu-ch'u-wu 一五除五 (One-five-take-five)	Meet 5, and place 10 above, the remainder is 1 ; From the number itself take 10, and return 5 to the place below.

𝍠 〇 𝍭 𝍧 𝍱 (1 0 5 8 9)	(2)	Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten) I-wu-ch'u-wu 一五除五 (One-five-take-five)	Meet 5 and place 10 above, the remainder is 5. From the number itself take 10, and return 5 to the place below and the remainder is 4.
𝍠 𝍠 𝍫 𝍤 𝍱 (1 1 4 5 9)	(3)	Wu-ssŭ-pei-tso-pa 五四倍作八 (Five-four-make-eight) Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten) Wu-chiu-ch'u-ssŭ-shih-wu 五九除四十五 (Five-nine-take-forty-five)	Change 4 the remainder and make it 8. Meet 5 and place 10 above this, the remainder is 8. As to the number itself, take 40 from it, and take 5 from the lower place, the remainder is 4.
𝍠 𝍠 𝍱 𝍣 𝍫 (1 1 9 4 4)	(4)	Wu-ssŭ-pei-tso-pa 五四倍作八 (Five-four-make-eight) Wu-pa-ch'u-ssŭ-shih 五八除四十 (Five-eight-take-forty)	Change 4 the remainder and make it 8. Take 40 from it. All is exhausted.

(Note: Of the above illustrations of the block-calculation, those below (2) have been added by this compiler according the explanations. They are not given in the original.)

Explanation: 6589÷55

(1) Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten)
I-wu-ch'u-wu 一五除五 (One-five-take-five)

(2) Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten)
I-wu-ch'u-wu 一五除五 (One-five-take-five)

(3) Wu-ssŭ-pei-tso-pa 五四倍作八 (Five-four-make-eight)
Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten)
Wu-chiu-ch'u-ssŭ-shih-wu 五九除四十五 (Five-nine-take-forty-five)

(4) Wu-ssŭ-pei-tso-pa 五四倍作八 (Five-four-make-eight)
Wu-pa-ch'u-ssŭ-shih 五八除四十 (Five-eight-take-forty)

```
          6  5  8  9
         -5
(1)   +1  0
         -1  0
            +5
     ----------------
      1   0  5  8  9
             5
            -5
(2)      +1  0
            -1  0
               +5
     ----------------
      1   1  4  5  9
                8
            -4
            +8
(3)            -5
            +1  0
               -4  5
     ----------------
      1   1  9  4  4
               -4
(4)            +8
                  -4  0
     ----------------
      1   1  9  8
```

As to the above calculations, considering the calculation (1) and (2), and such an instance as 99801 ÷ 999, it may be clear that these division methods were for block-calculation; for they would have been too difficult for abacus operations. (Note. Should the Ch'ung-kuei 撞歸 method be used here, the reckoning-rhymes used for the above operations would be in the following order. Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten), I-wu-ch'u-wu 一五除五 (one-five-take five); Fêng-wu-chin-i-shih 逢五進一十 (Meet-five-above-ten), I-wu-ch'u-wu 一五除五 (One-five-take-five); Chien-wu-ch'u-tso-chiu-wu 見五無除作九五 (See-five-no-dividend-make-ninety-five), Wu-chiu-ch'u-ssŭ-shih-wu 五九除四十五 (Five-nine-take-forty-five); Wu-ssŭ-pei-tso-pa 五四倍作八 (Five-four-make-eight), Wu-pa-ch'u-ssŭ-shih 五八除四十 (Five-eight-take-forty).

6. Development of the Kuei-ch'u-fa 歸除法 (Division Rhyme method)

The chiu-kuei 九歸 method evolved as a shortened method invented for division with one place divisor, through YANG Hui's days in the latter part of the Sung period, and by the Yüan period, it had established itself replacing the Shang-ch'u 商除 method, though not as an orthodox method, but as a common usual method.

The Suan-hsüeh-ch'i-mêng 算學啓蒙 (1299) by CHU Shih-chieh 朱世傑 contains this passage.

按古法多用商除，爲初學者難入，則後人以此法代之，卽非正術也

"I am of the opinion that as for the old method, people mostly used the Shang-ch'u 商除method. It is difficult for the beginner. Therefore, people now-a-days replace it by the division-rhyme method. But it is not an orthodox method."

In view of the fact that CHU Shih-chieh who calls this anu northodox method fails to mention the Shang-ch'u 商除 except in connection with extraction, it is probable that this new method had been considerably diffused by this time.

This book contains the Chiu-kuei-ko-chüeh 九歸歌訣 (Division by one-figure numbers accompanied by division-rhymes) resembling the modern, which will be described later on, but it like those books at YANG Hui's time, fails to give the rhyme "Ch'ung-kuei-huan-yüan 撞歸還原". However, in view of the fact that the Chiu-kuei-ch'u-fa-mên 九歸除法門 the First Volume, contains the lines:

法實相停九十餘，但遇無除還頭位

"When the divisor and the dividend are equal, make 90 and odd. If you meet with no dividend, recover the top number." Most probably the method was in use, though the precise phrase Ch'ung-kuei-huan-yüan 撞歸還原 does not appear.

The reason why the Shang-ch'u 商除 method, replacing the Kuei-ch'u 歸除 method, came to be extensively used, in my view, was because the rhymes of the Ch'ung-kuei-huan-yüan 撞歸還原 method came to be used extensively.

The extensive practice of the Kuei-ch'u 歸除 method may be known by another line of the above-cited poem which says: 流傳故泄眞消息 (As it comes down by tradition, the true history is not known).

Now, the Chiu-kuei-ko-chüeh 九歸歌訣 represented in this book run like this :—

I-kuei-ju-i-chin 一歸如一進 (Division by one, above-dividend)
Chien-i-chin-ch'êng-shih 見一進成十 (See-one-above-make-ten)
Êrh-i-t'ien-tso-wu 二一添作五 (Two-one-attach-make-five)
Fêng-êrh-chin-i-shih 逢二進一十 (Meet-one-above-ten)
San-i-san-shih-i 三一三十一 (Three-one-make-thirty-one)
San-êrh-liu-shih-êrh 三二六十二 (Three-one-make-sixty-two)
Fêng-san-chin-ch'êng-shih 逢三進成十 (Meet-three-above-ten)
Ssŭ-i-êrh-shih-êrh 四一二十二 (Four-one-make-twenty-two)
Ssŭ-êrh-t'ien-tso-wu 四二添作五 (Four-two-attach-make-five)
Ssŭ-san-ch'i-shih-êrh 四三七十二 (Four-three-make-seventy-two)
Fêng-ssŭ-chin-ch'êng-shih 逢四進成十 (Meet-four-above-make-ten)
Wu-kuei-t'ien-i-pei 五歸添一倍 (Division by five, attach double)
Fêng-wu-chin-ch'êng-shih 逢五進成十 (Meet-five-above-ten)
Liu-i-hsia-chia-ssŭ 六一下加四 (Six-one-add-four-down)
Liu-êrh-san-shih-êrh 六二三十二 (Six-two-make-thirty-two)
Liu-san-t'ien-tso-wu 六三添作五 (Six-three-attach-make-five)
Liu-ssŭ-liu-shih-ssŭ 六四六十四 (Six-four-make-sixty-four)
Liu-wu-pa-shih-êrh 六五八十二 (Six-five-make-eighty-two)
Fêng-liu-chin-ch'êng-shih 逢六進成十 (Meet-six-above-ten)
Ch'i-i-hsia-chia-san 七一下加三 (Seven-one-add-three-down)
Ch'i-êrh-hsia-chia-liu 七二下加六 (Seven-two-add-six-down)
Ch'i-san-ssŭ-shih-êrh 七三四十二 (Seven-three-make-forty-two)
Ch'i-ssŭ-wu-shih-wu 七四五十五 (Seven-four-make-fifty-five)
Ch'i-wu-ch'i-shih-i 七五七十一 (Seven-five-make-seventy-one)
Ch'i-liu-pa-shih-ssŭ 七六八十四 (Seven-six-make-eighty-four)
Fêng-ch'i-chin-ch'êng-shih 逢七進成十 (Meet-seven-above-ten)
Pa-i-hsia-chia-êrh 八一下加二 (Eight-one-add-two-down)
Pa-êrh-hsia-chia-ssŭ 八二下加四 (Eight-two-add-four-down)
Pa-san-hsia-chia-liu 八三下加六 (Eight-three-add-six-down)
Pa-ssŭ-t'ien-tso-wu 八四添作五 (Eight-four-attach-make-five)
Pa-wu-liu-shih-êrh 八五六十二 (Eight-five-make-sixty-two)
Pa-liu-ch'i-shih-ssŭ 八六七十四 (Eight-six-make-seventy-four)
Pa-ch'i-pa-shih-liu 八七八十六 (Eight-seven-make-eighty-six)
Fêng-pa-chin-ch'êng-shih 逢八進成十 (Meet-eight-above-ten)
Chiu-kuei-sui-shên-hsia 九歸隨身下 (Division by nine, add-dividend-down)
Fêng-chiu-chin-ch'êng-shih 逢九進成十 (Meet-nine-above-ten)

The rhymes are 36 in all. The Wu-kuei 五歸 and the Chiu-kuei 九歸, with some omissions, are very short.

7. Advent of the Ch'ung-kuei-huan-yüan 撞歸還原 rhymes.

The method of Ch'ung-kuei-huan-yuan 撞歸還原 represented in the poem in the *Suan-hsüeh-ch'i-mêng* 算學啓蒙 first appeared in the *Ting-chü-suan-fa* 丁巨算法 (1355). In the course of explaining problems, a line reading Ch'ung-kuei-chiu-shih-san 撞歸九十三 occurs. *The Suan-fa-ch'üan-nêng-chi* 算法全能集 by CHIA Têng 賈亨 (CHIA T'ung 賈通 according to the *Yung-lo-ta-tien* 永樂大典) admittedly published soon after this, includes under the title Ch'ung-kuei-fa 撞歸法 are the rhymes :—

Êrh-kuei-wei-chiu-shih-êrh 二歸爲九十二 (Division-by-two-make-ninety-two)
San-kuei-wei-chiu-shih-san 三歸爲九十三 (Division-by-three-make-ninety-three)
Ssŭ-kuei-wei-chiu-shih-ssŭ 四歸爲九十四 (Division-by-four-make-ninety-four)
Wu-kuei-wei-chiu-shih-wu 五歸爲九十五 (Division-by-five-make-ninety-five)
Liu-kuei-wei-chiu-shih-liu 六歸爲九十六 (Division-by-six-make-ninety-six)
Ch'i-kuei-wei-chiu-shih-ch'i 七歸爲九十七 (Division-by-seven-make-ninety seven)
Pa-kuei-wei-chiu-shih-pa 八歸爲九十八 (Division-by-eight-make-ninety-eight)
Chiu-kuei-wei-chiu-shih-chiu 九歸爲九十九 (Division-by-nine-make-ninety-nine)

As to the Huan-yüan 還原 method, however, we find only these lines :

歸有若是無除數，起一囘將元數施

" If after division, no dividend is left, take one from the dividend, and add the top number to the dividend)." No Huan-yüan 還原 rhymes are mentioned.

The *Hsiang-ming-suan-fa* 詳明算法 (1373), considerably influenced by this book, and explaining the difference between Ch'ung-kuei 撞歸 and Ch'i-i 起一, gives the rhyme Ch'i-i-huan-wu 起一還五, but with no complete explanation of it.

The first appearance of the whole rhymes of the Huan-yüan method was only in the *Chiu-chang-hsiang-chu-pi-lei-suan-fa-ta-ch'üan* 九章詳註比類算法大全 (1450) by WU Ching (Hsin-min) 吳敬(信民).

" Ch'ung-kuei-fa 撞歸法 (Note, 250÷29

29	250	
爲九十二 (make 92)	+72	Change 2 to 9 ; add 2 down
	970	9×9=81. Cannot be subtracted from 70 ; therefore
減一 (subtract 1)	−1	subtract 1, and
	870	
下還二 (add 2 down)......	+ 2	add 2 down
	890	

8×9 ....$-\ 72$ $\quad 8\times9=72$ is subtracted

818

$\therefore$ 8 is the quotient and 18 the remainder.

Êrh-kuei-wei-chiu-shih-êrh 二歸爲九十二

[Wu-ch'u-chien-i-hsia-huan-êrh 無除減一下還二 (No-dividend-subtract-one, add-return-two-down)]

San-kuei-wei-chiu-shih-san 三歸爲九十三

[Wu-ch'u-chien-i-hsia-huan-san 無除減一下還三 (No-dividend-subtract-one, add-return-three-down)]

Ssŭ-kuei-wei-chiu-shih-ssŭ 四歸爲九十四

[Wu-ch'u-chien-i-hsia-huan-ssŭ 無除減一下還四 (No-dividend-subtract-one, add-return-four-down)]

Wu-kuei-wei-chiu-shih-wu 五歸爲九十五

[Wu-ch'u-chien-i-hsia-huan-wu 無除減一下還五 (No-dividend-subtract-one, add-return-five-down)

Liu-kuei-wei-chiu-shih-liu 六歸爲九十六

[Wu-ch'u-chien-i-hsia-huan-liu 無除減一下還六 (No-dividend-subtract-one, add-return-six-down)]

Ch'i-kuei-wei-chiu-shih-ch'i 七歸爲九十七

[Wu-ch'u-chien-i-hsia-huan-ch'i 無除減一下還七 (No-dividend-subtract-one, add-return-seven-down)]

Pa-kuei-wei-chiu-shih-pa 八歸爲九十八

[Wu-ch'u-chien-i-hsia-huan-pa 無除減一下還八 (No-dividend-subtract-one, add-return-eight-down)]

Chiu-kuei-wei-chiu-shih-chiu 九歸爲九十九

[Wu-ch'u-chien-i-hsia-huan-chiu 無除減一下還九 (No-dividend-subtract-one, add-return-nine-down)]."

This book, in a chapter on Ho-t'u-shu-shu 河圖書數 contains a sentence to the effect that calculations may be conducted without using the abacus or reckoning-blocks; this shows that the abacus practice had been considerably diffused by this time. Then the *Ku-chin-suan-hsüeh-pao-chien* 古今算學寶鑑 (preface, 1524) by Wang Wên-su 王文素 includes the rhyme Chien-i-wu-ch'u-tso-chiu-i 見一無除作九一 (See-one-no-dividend-make-ninety-one), which had never before been recorded; this completed the Ch'ung-kuei 撞歸 rhymes.

The fact that this book explains the Chung-chiu-hsiang-ch'êng 衆九相乘 (Note, 999×999 etc.) by saying:

衆九相乘，用子甚多算盤子少，乘則不便

"The Chung-chiu-hsiang-ch'êng 衆九相乘 has a great deal to do with the beads. The abacus has not enough beads. In multiplication inconvenience is felt." Though the book contains no illustrations of the abacus, it is evident that the book is one on abacus mathematics.

From the foregoing, it may be seen that with the coming of books on abacus mathematics, beginning with thr *Ku-chin-suan-hsüeh-pao-chien* 古今算學寶鑑, if I cannot say whether the *Hsiang-ming-suan-fa* 詳明算法 was a book on abacus-mathematics or block-mathematics, the Ch'ung-kuei 撞歸 rhymes on i-kuei 一歸 came and the Kuei-ch'u method (Division method practised even to-day) came to be roughly completed.

I have previously stated that, in the *Chiu-chang-hsiang-chu-pi-lei-suan-fa-ta-ch'üan* 九章詳註比類算法大全, the Huan-yüan 還原 rhymes are introduced as a supplement to the description of the Ch'ung-kuei 撞歸 rhymes. An independent treatment of this apart from the Ch'ung-kuei 撞歸 rhymes dates from the *Shu-hsüeh-t'ung-kuei* 數學通軌 published in 1578 which was compiled by Ko Shang-chien 柯尙遷. This, as an extant book containing an illustration of the abacus, is preceded only by the *P'an-chu-suan-fa* 盤珠算法 (1573).

As to under what branches, division in those days was conducted, quotations will be made from the *Hsiang-ming-suan-fa* 詳明算法.

Chiu-kuei 九歸 (十以下單位分者用此法，先從首位大數算起，用因法還元)
(With numbers below tens, this method is used. Start calculation with the highest order. By means of one place multiplication, the number may be reduced to the original.)

Ting-shên-ch'u 定身除 (卽減法也，一十，一百，一千，一萬之類，但首位一數者以此法令之，從首位大數算起，用加法還元)
"This is the so-called chien method. Numbers like tens, hundreds, thousands, tens of thousands, only with one-place numbers on the highest place, this method may be operated. Operation may be started with the highest place. Using the chia method may be reduced to the original."

Kuei-ch'u 歸除 (二十以上至百千萬以上，首位有二三四五六七八九者，皆以此法分之，從首位大數算起，用乘法還元)
(In dividing numbers such as 2, 3, 4, 5, 6, 7, 8, 9 of a single place on the highest place, this method may be used. By means of the Chêng-fa 乘法 the numbers may be reduced to the original.) They are classified into three classes:

Namely,—
a. Those with one-place divisor Chiu-kuei 九歸
b. Those with a divisor above two places with 1 as its initial number Ting-shêng-ch'u 定身除
c. Those with a divisor of more than 2 places but not with 2. Kuei-ch'u 歸除

In cases with a divisor with 1, what we call Shêng-i-ch'u-fa 省一除法 is used. So it is not necessary to use the Ch'ung-kuei 撞歸 method of I-kuei 一歸.

Using reckoning-blocks in operating the Shêng-i-ch'u-fa 省一除法, from the very nature of a calculating instrument, may have been freer and the calculation simpler and quicker.

It may be said that, the progress of calculating instrument from the reckoning-blocks to the abacus and the advent of the Chʻung-kuei 撞歸 rhymes on I-kuei 一歸, between the last part of the 15th century and the beginning of the 16th century, played a prominent part in the development of calculation when, as a result of the perfection of the Kuei-chʻu-ko-chüeh 歸除歌訣 the division performed by the reckoning-blocks was now superseded by that performed by the reckoning-blocks.

8. Decline of the Shang-chʻu 商除 method

That division in ancient times was the Shang-chʻu method like that described in the *Sun-tzŭ-suan-ching* 孫子算經; and against this Shang-chʻu method there arose the Chiu-kuei 九歸 method as a shortened form in case the divisor is of one place, and this method, after Yang Hui 楊輝 came to be extended and applied to a divisor of more than two places;

That the Kuei-chʻu 歸除 method prior to the existence of the Chʻung-kuei-huan-yüan 撞歸還原 method, had little raison d'être as a method of division;

That, therefore, in such times the Chiu-kuei 九歸 method only had a raison d'être as a shortened form for division with a divisor of one place.

That, at the time of the *Suan-hsüeh-chʻi-mêng* 算學啓蒙, in spite of the fact that the Kuei-chʻu method was rejected as unorthodox method, as stated previously, the Kuei-chʻu 歸除 method when combined with the Chʻung-kuei-huan-yüan 撞歸還原 method, abruptly began to display its excellence as a worthy method of division.

Now, what became of the Shang-chʻu 商除 method branded as the orthodox method by Chu Shih-chieh 朱世傑?

The *Suan-hsüeh-chʻi-mêng* 算學啓蒙 includes not as a method of division by means of the Shang-chʻu method, but only as a calculative method necessary for extraction of roots.

The *Suan-fa-chʻüan-nêng-chi* 算法全能集 says:

其法較別，且如九歸但能分爲九分，定身除止能分一十百千萬令，歸除又只能分二三十以上有零之數，今商除一法却該三法之分，欲求其往捷，終不若前三法之疾，本不載此商除，以惑算者之心，然此法別有用處，又不容不載於其間.

"To compare these different methods: The Chiu-kuei 九歸 method only deals with division for the numbers from 1 to 9. The Ting-shên-chʻu 定身除 method deals with division for numbers with divisors with 1 on the top place, such as 11–19, 101–199, 1001–1999. The Kuei-chʻu 歸除 method deals for numbers with 2 more places divisors such as 21–99, 201–999, 2001–9999. Now, this method, the Shang-chʻu 商除 method combines all these three methods. But if you wish speedy calculation, it is not equal to the three former methods. Lest this method should confuse the student's mind, it has not been described in the book. However, as this method might be utilized elsewhere, it is impossible to ignore the method."

Again, in the *Hsiang-ming-suan-fa* 詳明算法, it is said:

商除者商量而除之，此一術亦兼九歸定身除歸除三法，旣通歸除不必學此，但開方則必商除，故不可廢其法.

"In the Shang-ch'u method, division is done by consideration. This one method also combines the three methods, namely, the Chiu-kuei 九歸, the Ting-shên-ch'u 定身除 and the Kuei-ch'u 歸除. If one is versed only in the Kuei-ch'u 歸除, one need not learn this. Only in extracting squares, this Shang-ch'u 商除 is absolutely necessary."

In this way, the Shang-ch'u method came to exist only as a calculative method necessary only for extraction of roots.

As the rise of mathematics among the common people in the Ming period was primarily prompted by practical calculation, it needed no such high-class calculation as extraction of roots, but learned and developed only what was needed; therefore the existence of the Shang-ch'u method became more and more faint.

(Note: A word may be necessary for the relationship between evolution and the Shang-ch'u method.

In the earliest period, the method of extraction of roots like the Shang-ch'u method described in the *Sun-tzŭ-suan-ching* 孫子算經, consisted of the way of putting the quotient (商) above the product (積.)

In later times, however, as described in the *Hsiang-ming-suan-fa* 詳明算法 "商惚分排兩位居" which means the quotient (商) and the dividend (惚) are distributed to two positions with by outting the quotient to the left of the dividend, the operation was shifted so as to favor the view that the abacus appeared in order to supersede the 籌算 (block-mathematics).

Furthermore, as represented in the *Suan-fa-t'ung-tsung* 算法統宗, abacus mathematics advanced so far as to solve multiplication, division, extraction of roots, it became impossible to say that so for as extraction of roots was concerned, they had to avail themselves of the Shang-ch'u method. Therefore, attention must be paid to the fact that the Shang-ch'u method was not always for extraction of roots.)

9. Conclusion

The foregoing is a general survey, up to approximately the middle of the Ming period, of division methods (the Shang-ch'u-fa 商除法 and the Kuei-ch'u-fa 歸除法) in China, and a history of the establishment of the Kuei-ch'u-fa.

A study of the group of those mathematical books would show a shift from those on block-mathematics to those on the abacus ensuing the completion of the Kuei-ch'u-ko-chüeh 歸除歌訣, which introduced an era of popularity for the abacus in the Ming period, and after this, it was thought that the Kuei-ch'u-fa 歸除法 was the only method of division for the abacus ; and as the construction of the abacus fitted the Kuei-ch'u method so perfectly as if it had been constructed for operating the Kuei-ch'u method, the Kuei-ch'u method and the abacus were considered as related to each other, and even in later times nobody ever doubted their relationship. The idea of connecting the advent of the Kuei-ch'u-ko-chüeh 歸除歌訣 with the origin of the abacus would be proved to be entirely inadequate if we carefully investigated :—(a) the purpose for which the Kuei-chu-ko-chüeh came into being ; (b) how it was revised and improved ; (c) how it served to heighten the value of mathematics ; and (d) how it succeeded in expelling the Shang-ch'u 商除 method.

Chapter III

The Theory that Holds that the Abacus was Introduced to Replace Suan-mu 算木 (Reckoning-Blocks)

This is the theory that holds that in place of the wooden blocks used in suan-mu 算木 calculation, the abacus with beads came to replace them, that the abacus appeared to replace the suan-mu as a reckoning instrument.

The fact that the numerical expression of the reckoning-blocks contains an idea of five and the abacus likewise the idea of five beads may lead one to favor this view. However, the reckoning-blocks and the abacus are not the only cases where the idea of five is made use of. The fact supplies no definite ground for the argument.

Even if this view were taken as an adequate view, the fact that since the *Cho-kêng-lu* 輟耕錄 (1366) by T'ao Tsung-i (Nan-ts'un) 陶宗儀 (南村) contains the expression suan-p'an-chu 算盤珠 and the common saying on "the three beads", and the *Ching-hsiu-hsin-shêng-wên-chi* 靜修先生文集 (1279) contains a poem on the abacus, it may be considered that the practice of the abacus must have been diffused considerably by these times, and therefore the date of the introduction of the abacus must go further back.

However, even in those days, as has been discussed in Chapter II, the calculations in multiplication and division were classified as folllows :

		除法 (Division)	乘法 (Multiplication)
(1)	法一位 (one-place multiplier, or divisor)	九歸法 (Chiu-kuei-fa)	因法 (Yin-fa)
(2)	法二位以上		

	(more han one-place multiplier or divisor)		
a.	法 首 1 when the top number is 1	身外減法 (Shên-wai-chien-fa)	身外加法 (Shên-wai-chia-fa)
b.	法 首 2–9 when the top number is from 2 to 9.	歸除法 (Kuei-ch'u-fa)	留頭乘法 (Liu-t'ou-ch'êng-fa)

In case the top number of the multiplier or divisor was 1, before the Kuei-ch'u-fa had not yet been adopted, it had to be operated by the Shên-wai-chien-fa 身外減法. Under such circumstances, over against the block-reckoning which was far easier in analysing numbers than the abacus which with five beads fixed on the bar was much more restricted than the blocks,—especially when the abacus was to be a thing for the common people who more than anything depended upon practical mathematics, it would be quite questionable to hold that the abacus was invented as a calculative instrument to serve as the substitute for the reckoning-blocks, and that it was developed from the idea ofthe reckoning-blocks.

Chapter IV.

The Theory that Identifies the Advent of the *P'an-chu-chi* 盤珠集 and *Tsou-p'an-chi* 走盤集 with that of the Abacus

This view, on account of the appearance of the titles *P'an-chu-chi* 盤珠集 and *Tsou-p'an-chi* 走盤集 mentioned among the Suan-hsüeh-yüan-liu 算學源流 (Mathematical sources) at the end of the *Suan-fa-t'ung-tsung* 算法統宗 *Preface 1592*, regardsthem as the oldest books on abacus calculation and incidentally as the time of the advent of the abacus.

The *Suan-hsüeh-yüan-liu* 算學源流 in the *Suan-fa-t'ung-tsung* 算法統宗 will be cited in the following.

Printed in the 7th year of Yüan-fêng 元豐 of Sung (1084) 刊十書人秘省, 又刻于汀州學校 (reprinted at T'ing-chou-hsüeh-chiao 汀州學校.)

Huang-ti-chiu-chang 黃帝九章, *Chou-pi-suan-ching* 周髀算經, *Wu-ching-suan-fa* 五經算法. *Hai-tao-suan-ching* 海島算經, *Sun-tzŭ-suan-fa* 孫子算法, *Chang-ch'iu-chien-suan-fa* 張丘建算法, *Wu-ts'ao-suan-fa* 五曹算法, *Ch'i-ku-suan-fa* 緝古算法, *Hsia-hou-yang-suan-fa* 夏侯陽算法, *Suan-shu-shih-i* 算術拾遺.

Since Yüan-fêng 元豐 (1078–1085), Shao-hsing 紹興 (1131–1162) and

Chun-hsi 淳熙 (1178–1189), many editions were published. And those who had seen or heard of them wrote books are,

I-ku-kên-yüan 議古根源, *I-ku-suan-fa* 益古算法, *Chêng-ku-suan-fa* 證古算法, *Ming-ku-suan-fa* 明古算法, *Pien-ku-suan-fa* 辨古算法, *Ming-yüan-suan-fa* 明源算法, *Chin-k'o-suan-fa* 金科算法, *Chih-nan-suan-fa* 指南算法, *Ying-yung-suan-fa* 應用算法, *Ts'ao-t'ang-suan-fa* 曹唐算法, *Chia-hsien-chiu-chang* 賈憲九章, *T'ung-wei-chi* 通微集, *T'ung-chi-chi* 通機集, *P'an-chu-chi* 盤珠集, *Tsou-p'an-chi* 走盤集, *San-yüan-hua-ling-ko* 三元化零歌, *Ling-ching* 鈐經, *Ling-shih* 鈐釋.

In the eras of Chia-ting 嘉定, Hsien-ch'un 咸淳, and Tê-yu 德祐, several books were reprinted.

Hsiang-chieh-huang-ti-chiu-chang 詳解黄帝九章, *Hsiang-chieh-jih-yung-suan-fa* 詳解日用算法, *Chêng-ch'u-t'ung-pien-pên-mo* 乘除通變本末, *Hsü-ku-chai-ch'i-hsiang-ming-suan-fa* 續古摘奇算法, —all these are found in Yang Hui's *Chai-ch'i* 楊輝摘奇.

Hsiang-ming-suan-fa 詳明算法, *Chiu-chang-t'ung-ming-suan-fa* 九章通明算法, *Chih-ming-suan-fa* 指明算法, *Chiu-chang-pi-lei-suan-fa* 九章比類算法, *Suan-hsüeh-t'ung-yen* 算學通衍, *Chiu-chang-hsiang-chu-suan-fa* 九章詳註算法, *Ch'i-mêng-fa-ming-suan-fa* 啓蒙發明算法, *Ma-chieh-kai-chêng-suan-fa* 馬傑改正算法, *Chü-ku-suan-shu* 句股算術, *Chêng-ming-suan-fa* 正明算法, *Suan-li-ming-chieh* 算理明解, *Chung-ming-suan-fa* 重明算法, *Ting-chêng-suan-fa* 訂正算法, *Tsê-yüan-hai-ching* 測圓海鏡, *Hu-shih-hsien-shu* 弧矢弦術, *Suan-lin-pa-ts'ui* 算林拔萃, *I-hung-suan-fa* 一鴻算法, *Yung-chang-suan-fa* 庸章算法, *Chiu-chang-hsiang-t'ung-suan-fa* 九章詳通算法.

So it happens that the *P'an-chu-chi* and the *Ts'ou-p'an-chi* are mentioned among the books reprinted since Yüan-fêng 元豐, Shao-hsing 紹興 and Ch'un-hsi 淳熙, in different editions written by those who had seen or heard of them.

Considering the expression "seen or heard of", it is not clear whether Ch'eng Ta-i 程大位, the author of the *Suan-fa-t'ung-tsung* actually saw this book.

According to this, the *P'an-chu-chi* and the *Tsou-p'an-chi*, along with the *Chih-nan-suan-fa* 指南算法 are found among the books printed since Yüan-feng, Shao-hsing, and Ch'un-hsi, and the *Jih-yung-suan-fa* 日用算法 *Chêng-chu-t'ung-pien-pên-mo* 乘除通變本末 by Yang Hui 楊輝 is among the books reprinted during the eras years of Chia-ting 嘉定 (1208–1223), Hsien-ch'un 咸淳 (1265–1274) and Tê-yu 德祐 (1275).

As stated in Section II, judging from the evolution of the Kuei-ch'u-fa 歸除法 as represented in the *Chih-nan-suan-fa* 指南算法 or the *Yang-hui-suan-fa* 楊輝算法 it could not be imagined that, chronologically speaking, the *P'an-chu-chi* or the *Tsou-p'an-chi*, had adopted in its calculative operations multiplication and division methods like the Kuei-chu-fa 歸除法 or Liu-t'ou-ch'êng-fa 留頭乘法 as the abacus books of later times dated after the middle of the Ming period.

Simply because of the resemblance in the ideographs for the abacus or of the possibility of associating it with the abacus, it could not be possible to determine the date of the advent of the abacus. If inferred from this kind of theory, relationships with the abacus might be established from,

Li Shao-ku 李紹穀: *Ch'iu-i-chih-chang-suan-shu-hsüan-yao* 求一指掌算術玄要
Hsu Jên-mei 徐仁美: *Ts'êng-ch'êng-hsüan-i-suan-fa* 增成玄一算法 (or *Ts'êng-ch'êng-hsüan-i-fa* 增成玄一法)
Jên Hung-chi 任弘濟: *I-i-suan-fa-wên-ta* 一位算法問答 *Fa-suan-k'ou-chüeh* 法算口訣 (or *Suan-fa-k'ou-chüeh* 算法口訣)
Wu-ts'ao-ch'êng-ch'u-chien-i-chieh-li-suan-fa 五曹乘除見一捷例算法
Ch'iu-i-suan-fa 求一算法
Chieh-fa-ch'iu-i-hua-ling-ko 解法求一化零歌

Among the mathematical books included in the *Sung-shih-i-wên-chih* 宋史藝文志, *Ch'ung-wên-tsung-mu* 崇文總目 in the *Chung-kuo-suan-hsüeh-shih* 中國算學史 (p. 87 ff.) by Li Yen 李儼 and the *Suan-fa-hsü-shuo* 算法序說, *Ch'êng-ch'u-suan-li* 乘除算例 included in the *Pi-shu-shêng-hsü-pien-tao-ssŭ-ku-shu-mu* 秘書省續編到四庫書目 officially compiled during Shao-hsing 紹興 (1131–1162) of the Sung dynasty.

If the *P'an-chu-chi* and the *Tsou-p'an-chi* had been abacus books, the calculative methods adopted by the books should have been the Chin-ch'an-t'o-ku-fa 金蟬脫殼法 and the Êrh-chü-shih-chüeh 二字口訣 referred to in the following chapter.

Now that these two books have been lost sight of, it could only be supposed that they were exactly like the *Cho-kêng-lu* 輟耕錄 widely diffused, and the abacus which was especially handy in calculations in addition and subtraction came to be slowly but extensively diffused among the people at large. It would seem too far-fetched a view to identify the advent of these books with that of the abacus.

Chapter V

The Theory that Holds that the Abacus was Introduced from Rome

In the Introductory, it has been pointed out that the extreme resemblance in the construction of the abacus introduced in the *Shu-shu-chi-i* 數術記遺 with the grooved Roman abacus and the economic and ideological significance which supports them make us infer that the Chinese abacus came from Rome, or otherwise it is in extremely intimate relations with the Roman. And as to the trade conducted between China and Rome in those days, the present writer has treated it in full detail in his *Chūgoku Soroban Oboegaki* 中國算盤覺書 (Notes on the Chinese Abacus), *Keizai Shūshi* 經濟集志, published by the Nihon University, Department of Economics.

In the present section the issue will be explained from the standpoint of calculation.

1. The origin of the modern abacus calculation method

Calculation by means of the reckoning-blocks practised in ancient China, as described in Chapter II, began with that of multiplying and dividing the dividend, divisor or multiplier, and quotient put down in three columns, the top, the middle, and the bottom, which is described in the *Sun-tzŭ-suan-ching* 孫子算經. This was advanced toward simplification until every simplified method such as Shên-wai-chia-fa 身外加法, Shên-wai-chien-fa 身外減法, Chüan-ch'êng 損乘, and for division the Kuei-ch'u 歸除 method and for multiplication the Liu-t'ou 留頭 method was adopted. All these changes were materialized by means of reckoning-blocks.

Now beginning with the ancient books included in the *Suan-ching-shih-shu* 算經十書, and containing with the *Yang-hui-suan-fa* 楊輝算法, *Suan-hsüeh-ch'i-mêng* 算學啓蒙, *Suan-fa-ch'üan-nêng-chi* 算法全能集, *Ting-chü-suan-fa* 丁巨算法, *Hsiang-ming-suan-fa* 詳明算法,—all the groups of the reckoning-block books from ancient times, through Sung, Yüan, and early Ming periods, our study will show all these changes in calculation.

After the middle of the Ming period we notice the appearance of the groups of abacus calculation books. Every one of them describe multiplication and division by means of the reckoning-blocks—like their predecessors, expound the Kuei-ch'u-fa 歸除法, the Liu-t'ou-ch'êng-fa 留頭乘法, and every other simplified calculation.

As to a sudden diffuse of the abacus after the middle part of the Ming period several reasons might be offered. Speaking from the standpoint of calculation, however, the ancient reckoning-block method of calculation by putting down numbers in three columns, the top, the middle, and the bottom was so simplified that calculation became possible by arranging side by side the dividend and divisor or multiplier and so developed that calculation was smoothly conducted by use of a calculator named abacus, and that it became possible to shift the Kuei-ch'u 歸除 method and the Liu-t'ou 留頭 method upon the abacus. The instrument became diffused and its characteristics, its simplicity as an calculator, and its absolute advantage as addition and subtraction calculator came to be recognized. This certainly was a chief reason for its replacing the reckoning-blocks during the middle part of the Ming period which emphasized practical mathematics.

In other words, the abacus which had been used from ancient times, after the middle part of the Ming period, suddenly became a favorite of the day when the people learned thoroughly to assimilate the calculative method of the reckoning-blocks and succeeded to shift it upon the abacus.

(Note: This may be demonstrated by the *Chiu-chang-hsiang-chu-pi-lei-suan-fa-ta-ch'üan* 九章詳註比類算法大全, and the *Ku-chin-suan-hsüeh-pao-chien* 古今算學寶鑑.)

2. Abacus calculation in the earliest stage

Success in shifting the Kuei-ch'u 歸除 method and the Liu-t'ou 留頭 method for calculating by means of the reckoning-blocks was the principal reason for the popularity of the abacus, as already stated.

Now, prior to this time, how had the abacus been used for multiplication and division? Now that most of the mathematical books from the *Suan-ching-shih-shu* 算經十書 to the *Yang-hui-suan-fa* 楊輝算法 have been lost sight of the full details could not be known. However, multiplication by means of the abacus prior to the invention of the Kuei-ch'u method and the Liu-t'ou method is described in the abacus calculation books written at the last part of the 16th century. It was presumably conducted by primitive calculative methods called Chin-ch'an-t'o-ku-fa 金蟬脫殼法 and Êrh-chü-shih-chüeh 二句字訣.

Comparison of the above-mentioned mathematical books on the reckoning-blocks, prior to the middle part of the Ming period, and the later abacus books, with reference to various calculative methods, would show a difference—a characteristic—.

Such abacus books of the Ming period as the *P'an-chu-suan-fa* 盤珠算法 (1573), the *Shu-hsüeh-t'ung-kuei* 數學通軌 (1578), and the *Suan-fa-t'ung-tsung* 算法統宗 (with the Preface, 1592) which all carry illustrations of the abacus, in addition to the Kuei-ch'u method and the Liu-t'ou method of multiplication and division described in the reckoning-block books, describe the Chin-ch'an-t'o-ku-fa 金蟬脫殼法 and the Êrh-chü-shih-chüeh 二句字訣 which are supposed to represent the most primitive method of multiplication and division by means of progressive addition and subtraction.

(Note: The ancient division method, namely the Shang-ch'u 商除 method is quite briefly described only in the *Suan-fa-t'ung-tsung* 算法統宗.)

This primitive method of multiplication and division by means of progressive addition and subtraction is not represented in the books on the reckoning-blocks. It is found in the *Chiu-chang-hsiang-chu-pi-lei-suan-fa-ta-ch'üan* 九章詳註比類算法大全 which makes the transition from the group of books on the block-mathematics to the group of books on abacus mathematics, but not in the *Ku-chin-suan-hsüeh-pao-chien* 古今算學寶鑑.

As the Chin-ch'an-t'o-ku-fa 金蟬脫殼法 and the Êrh-chü-shih-chüeh 二句字訣 thus become an issue, they may be introduced here.

a. The Chin-ch'an-ch'êng-fa-i 金蟬乘法義 (Explanation of Chin-ch'an multiplication)

不用九字相生之數，但置一箇原法，又置一箇倍法，只云除二加倍去一還元，又云起一

如原法加之，起二如倍法加之，如米一石價錢六錢四分，將米爲實，以六錢四分爲法，又別置一箇倍法，得一兩二錢八分，呼云起一加六四，起二加一二八，只此二句以代乘法，皆從實後置數，次第因而加之，又云，呼如加隔位，言十加次位，不疊本位也

"No multiplication rhyme is used. Only put down one original multiplicand and only one doubled multiplicand. If you subtract 2, add to a doubled multiplicand. If you take off 1, you return the original multiplicand. And if you move up 1, add it as in the original multiplicand.

If the price of one hu 斛 of rice is 6 ch'ien 錢 and 4 fên 分, the rice is put down as multiplicand 實, and 6 ch'ien and 4 fên as multiplier, and another double multiplier is put down, and one liang 兩 two ch'ien 錢 and 8 fên 分 is obtained. The rhyme for this says :—Ch'i-i-chia-liu-ssŭ 起一加六四 (Take-one-add-sixty-four,) Ch'i-êrh-chia-i-êrh-pa 起二加一二八 (Take-two-add-one-hundred-twenty-eight). These two rhymes will replace the method of multiplication. Now, put the number below the quotient and multiplying the number successively, add the numbers. When ju 如 (itself) is called, add to the place above one, and shih 十 (ten) is called, add to the next highest place. Don't put it on the number itself."

Explanation 1. To multiply a number by 64, put down as multipliers 64 and $2 \times 64 = 128$, and for each 1 taken from the multiplicand, add 64 to the product and for 2 taken from the multiplicand, add 128 to the product; thus continue operation in succession.

Explanation 2. As for $7 \times 64 = 448$

實 multiplicand		積 product	法 multiplier
	7		64 and 128
Take 2 and add 128	−2		
		128	
,,	−2		
		128	
,,	−2		
		128	
Take 1 and add 64.........	−1		
		64	
	0	448	

b. The Chin-ch'an-kuei-ch'u-fa-i 金蟬歸除法義 (Explanation of the Chin-ch'an division)

不用歸因之數，但置一箇原法，又置一箇半法，只云，進二除倍，添一還原，滿去過身一，折半當身五，又云，進一如原法除之，進五如半法除之．

如銀六錢四分買米一石，將銀爲實，以六錢四分爲法，又別置一箇半法，得三錢二分，呼云進一除六四，進五除三二，只此二句以代歸除，皆從實前置數，次第除之，又云進五身前位，進一前隔位，庶不疊本位

"Neither the Chiu-kuei-chü 九歸句—the division rhymes nor the multiplication rhymes are employed. Only an original divisor is put down with another in a halved divisor. You recite : " Advance 2, and subtract twice as many. Add 1 and return 1 to the original." You say : when the position is 1 above itself, and halving is equal to five of itself. As to advance 1 as in the original divisor, is equal to subtracting it, and as the halved dividor is called " Subtracting it ".

To buy one hu 斛 of rice with 6 ch'ien 錢 4 fên 分. The money is put down as the quotient, with 64 fên as the divisor. At the same time, a halved divisor, 32 fên. You recite : " For every one you advance, you take away 64 ; and for every 5, you advance, you take away 32." These two rhymes take the place of the division rhymes. In all cases, the number is put before the dividend, and subtracted successively. Another rhyme says : " In advancing 5, it is desireable that one should advance 1 on the place above itself and not pile it on itself."

As for $576 \div 64 = 9$—

For dividing 576 by 64, the divisors are put as 64 and 32 (a half) ; as you subtract 64 from the dividend, you add 1 to the result, and as you further subtract 32, you add 05 to the result ; you may calculate the rest successively.

		實 dividend	法 divisors
Subtract 32		5 7 6	64 and 32
Advance 5		−3 2	
	5		
Subtract 64	5	2 5 6	
Advance 1		−6 4	
	1		
Subtract 64	6	1 9 2	
Advance 1		−6 4	
	1		
Subtract 64	7	1 2 8	
Advance 1		−6 4	
	1		
Subtract 64	8	6 4	
Advance 1	1	−6 4	
	9	0	

The foregoing is from the *Shu-hsüeh-t'ung-kuei* 數學通軌.

c. The Êrh-chü-shih-chüeh 二句字訣 (Coupled reckoning rhymes).

有除隔位進，無除挨身進，隔一位除也（只用一原法，而無倍折數）但，因乘從實尾位起除一，隔一位而加原法數也，但，歸除從實前過一位起，亦隔一位而除原法數也，惟除實盡是數

'If there is a dividend, advance to the position skipping the one above it ; if no dividend, advance to the place immediately above it ; divide on the place skipping one place using the only original divisor, and with no number for halving or doubling. But in multiplication, conduct it successively from the bottom number of the dividend. In division, put 1 as the dividend on the position immediately above and divide the number of the original divisor skipping one place. Repeat this operation. When the dividend is exhausted, you will have obtained the answer."

Explanation 1. The idea is similar to that of the Chin-ch'an-t'o-ku-fa; only neither halving nor doubling is used, but only progressive subtraction by means of the original divisor.

Explanation 2.

$7 \times 64 = 448$

	multiplicand 實	
	7	
Subtract 1 and add 64	−1	64
	6	64
,,	−1	64
	5	128
,,	−1	64
	4	192
,,	−1	64
	3	256
,,	−1	64
	2	320
,,	−1	64
	1	384
,,	−1	64
	0	448

$320 \div 5 = 64$

		Dividend 實
		320
Subtract 64 and add 1	+1	−64
	1	256
,,	+1	−64
	2	192
,,	+1	−64
	3	128
,,	+1	−64
	4	64
,,	+1	−64
	5	0

The above are from the *Suan-fa-t'ung-tsung* 算法統宗. The division method is represented in the *Kaisanki* 改算記 (Printed in Japan, 1659).

When the books on abacus mathematics explain such excellent calculative methods as the Kuei-ch'u-fa 歸除法 and the Liu-t'ou-ch'êng-fa 留頭乘法 which are borrowed from the books on block—mathematics, why should they mention such primitive multiplication and division methods as the Chin-ch'an-t'o-ku-fa 金蟬脫殼法 and the Êrh-chü-shih-chüeh 二句字訣 which are operated by means of progressive subtraction and are never mentioned in the books on block-mathematics? It must be admitted before anything else that these calculative methods could not be independent of the abacus.

This must mean that, though the books on abacus mathematics of the Ming period borrowed the Kuei-ch'u-fa and 歸除法 the Liu-t'ou-ch'êng-fa 留頭乘法 and adopted them as the multiplication and division methods for the abacus, they had to include the primitive methods the Chin-ch'an-t'o-ku-fa and the Êrh-chü-shih-chüeh which had become so diffused among the people prior to the Ming period.

Ko Shang-ch'ien 柯尚遷, under Fang-t'ien-chang 方田章 in his *Shu-hsüeh-t'ung-kuei* 數學通軌 (1578) says:

或用二四歸除，或用四歸六歸各一囘，或用三八歸除，又或用五因六歸各一囘，又或用金蟬法...

"In place of dividing by 24, sometimes division by 4 and division by 6 are used each once (4×6 = 24) ; sometimes division by 3 and division by 8 are used (3×8 = 24) ; or sometimes multiplication by 5 and division by 6 each once ; or the Chin-ch'an-fa 金蟬法 is used."

3. Relationship with the Roman abacus

It has been pointed out that from the fact that the books on abacus mathematics published during the middle part of the Ming period, while borrowing the Kuei-ch'u method and the Liu-t'ou-ch'êng method which are the multiplication and division methods of block-mathematics, mention apart from them the Chin-ch'an-t'o-ku-fa and the Êrh-chü-shih-chüeh, progressive addition and subtraction were the very calculative methods which had been used for the abacus prior to the Kuei-ch'u and Liu-t'ou-ch'êng methods. In the present section, a few more remarks will be made in this connection.

Accounts of the abacus in Chinese bibliography, as has been introduced in the Introductory, occur in the *Shu-shu-chi-i* 數術記遺 written by Hsü Yo 徐岳 (the last part of the 2nd century) of the last part of the Latter Han period, and annotated by Chên Luan 甄鸞 (the middle part of the 6th century) of the Northern Chou.

The phrase chu-suan 珠算 occurs in this ; and the Notes by Chên Luan the represents the prototype of the modern abacus.

For a long time after this, there was no record whatever of the abacus. In the explanatory notes in the *Suan-ching* 算經 by Hsieh Ch'a-wei 謝察微 of the Sung period, the following words are given.

Chung 中 (middle) 算盤之中 (the centre of the abacus.)

The exact date of this book is not known, but probably of the Sung dynasty (960–1279).

The *Ching-hsü-hsien-shêng-wên-chi* 靜修先生文集 (Bk. 5) by Liu Yin 劉因 of the Yüan period (1248–1293) contains a five-word quartrain on the abacus. The book is said to have been written in 1279.

Then under the item Ching-chu 井珠 in the *Cho-kêng-lu* 輟耕錄 (1279) there occurs the account of the joke of "the three beads."

After the middle part of the Ming period, the phrase abacus is found in the following books on abacus mathematics.

The *Chiu-chang-hsiang-ch'u-pi-lei-suan-fa-ta-ch'üan* 九章詳註比類算法大全 (1450).

The *Ku-chin-suan-hsüeh-pao-chien* 古今算學寶鑑 (Preface, 1524) ; and the mathematical books after the last part of the Ming peiod, beginning with the *P'an-chu-suan-fa* 盤珠算法 (1573), all carry the illustrations of the abacus.

Thus the accounts of the abacus are so extremely scarce, and for this prolonged blank period, no bibliographical data whatever are found.

Let us now turn to the grooved Roman abacus which was almost contemporaneous and similar in construction with the abacus represented in the *Shu-shu-chi-i* 數術記遺 and that in a period during which the two countries traded with each other. Let us compare the two abacuses from the standpoint of calculation.

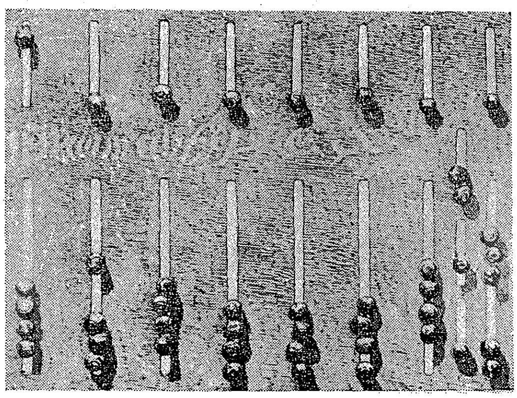
Concerning the Roman abacus, Florian CAJORI : *A History of Elementary Mathematics, with hints on methods of teaching*, revised and enlarged edition, New York, 1917 (translated into Japanese and annotated by Dr. Kinnosuke OGURA 小倉金之助) remarks : " In abacus calculation multiplication or division was conducted by repeating addition or subtraction."

René TATON, the French scholar : " *Histoire du calcul* ", (translated by Ken KOBORI 小堀憲, 1951) says to this effect. " The calculative theory of the abacus school is a product of studying the elementary Egyptian calculative operations,— namely addition, subtraction, doubling, halving, multiplication and division.— Addition and subtraction were operated on the abacus. In those days apis were used, so that the number of the check cards had decreased.

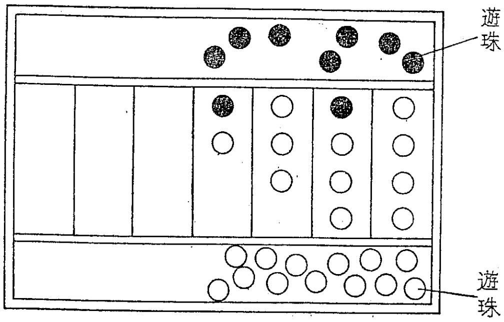
Doubling is a simpled way of adding two equal numbers. This was begun with the right end column. It must be remembered that to move the first column, the second column, etc. up to the left by one column meant multiplying by 10, by 100, etc......., ...Multiplication amounted to doubling, multiplying by 10, multiplying by 100, etc.......and to conducting addition in succession. These operations may be managed so easily by using the abacus.

For example, multiplying 423 by 47 will amount to conducting the following operations :

$$423\times7 = 423+(423\times2)+(423\times2)\times2 = 2{,}961$$
$$423\times40 = [(423\times2)\times2]\times10 = 16{,}920$$
$$423\times47 = 423\times40+423\times7 = 2{,}961+16{,}920 = 19{,}881$$



The Roman Abacus



The Abacus with mobile beads in the top and bottom compartments described in the Shu-shu-chi-i as pictured by this writer

Division amounted to a repetition of subtraction operations, and extraction of roots amounted to a repetition of analysis operations." Now, this method is exactly similar to the Chin-ch'an-t'o-ku-fa 金蟬脫殼法 of China already referred to. Therefore, considering the following facts :

a. Resemblance in construction between the abacus described in the *Shu-shu-chi-i* 數術記遺 and annotated by CHÊN Luan 甄鸞 and the grooved Roman abacus. (See the illustrations,)

b. Trade then carried on between China and Rome ;

c. Resemblance in operation between multiplication and division on the Roman abacus and that of ancient China on the abacus ;

d. Existence of the traces of calculation by the system of reckoning by 5's, for instance, in the Roman numerals :—

(six)......VI, 5+1 ; (seven)......VII, (5+2) ; (eight)......VIII, 5+3 ; (four)......IV, 5−1 ;

the present writer is of the opinion that the Chinese abacus either was imported from Rome, or is most closely connected with the Roman.

4. Inference of the history of the abacus prior the middle part of the Ming period.

The abacus which is inferred to have been imported from Rome in the course of the trade between the two countries, in the face of the feudal society, and in the face of the highly developed calculative art. The block-reckoning which was capable of calculating not only the simplest addition, subtraction, multiplication, division, but also quadric, cubic, or even simultaneous equations,—the abacus, a low-grade calculative instrument which was capable of calculating the multiplication and division operated by means of progressive addition and subtraction, deserved no notice on the part of the then receivers.

And probably this was the very reason why the common people more or less lacking the knowledge of calculation, came to use the abacus as a calculator primarily for addition and subtraction, and secondarily for simpler multiplication and division. For this reason, therefore, in China the abacus did not come up into the foreground, but remained in obscurity. This accounts for the prolonged blank periods.

With the development of economics, activities of the tradesman classes became the more lively. The abacus which they had for long periods cherished as a calculator among themselves, due to an accumulation of causes such as opportunities for the rise of practical mathematics the expansion of economic organizations, the advance of block-calculation, etc., succeeded in introducing the Kuei-ch'u method and the Liu-t'ou-ch'êng method of the reckoning-block mathematics into its operations systematized the method of head-operation (rhymes for 5's and rhymes for 10's which in the *Chiu-chang-hsiang-chu-pi-lei-suan-fa-ta-chüan* 九章詳註比類算法大全 are called Ch'i-wu-chüeh 起五訣 (Make-five-rhymes), P'o-wu-chüeh 破五訣, (Cancel-five-rhymes), Ch'êng-shih-chüeh 成十訣 (Make-ten-rhymes), P'o-shih-chüeh 破十訣 (Cancel-ten-rhymes) and which are explained, after the *P'an-chu-suan-fa* 盤珠算法, as Shang-fa 上法 (Addition method), and T'ui-fa 退法 (Subtruction method) and also called " chia-chien-chiu-chiu 加減九九 (addition and subtraction rhymes), and, to assist understanding supplied illustratins on each page, and further, abolishing the numerical notations adopted by the block-mathematics books and the early abacus books, which read 𝍠, 𝍡, 𝍢, 𝍣, 𝍤, 𝍥, 𝍦, 𝍧, 𝍨 and adopting the system of numerals called An-ma 暗馬 numerals which read 〡, 〢, 〣, 〤, 〥, 〦, 〧, 〨. 〥 or sometimes written ㄢ and thereby popularized itself among the common people.

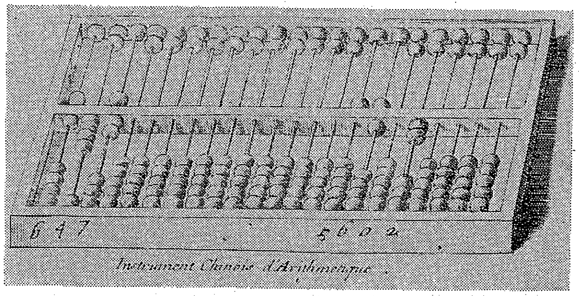
Abacus calculation became so simple and handy that block-calculation could not rival and as it reached a high stage at which most advanced multiplication and division could be operated, the abacus, it would seem, suddenly began to display its excellency as a calculating instrument, and with one bound established its popularity as a favorite of the age.

In spite of this fact, the tradesman classes unacquainted with the advanced Kuei-ch'u method or the Liu-t'ou method still exercised multiplication and division availing themselves of the old-fashioned Chin-ch'an-t'o-ku-fa 金蟬脫殼法 method or the Êrh-chü-shih-chüeh 二句字訣, the writers of the abacus books had to include these methods as well in the course of describing the different methods of calculation.

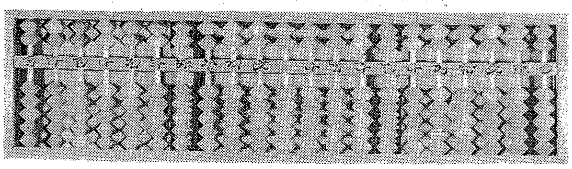
Chapter VI

CONCLUSION

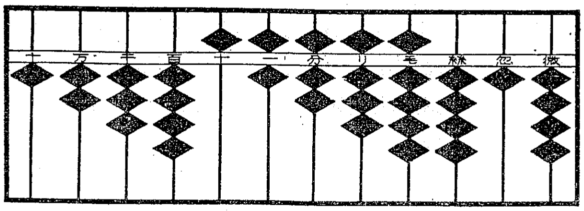
In the foregoing the present writer has investigated the several views as to the advent of the abacus with special reference to the methods of calculation (especially, division) and has expressed my criticism on these views in the respective sections. Speaking from the standpoint of calculation, as has been previously stated, I need not modify my view that the Chinese abacus was introduced from Rome.



The Chinese Abacus



The Japanese Abacus (19th Century)



The Japanese Abacus (20th Century)