History of Instrumental Multiplication and Division in China—from the Reckoning-blocks to the Abacus

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1. Introductory

The country in the Orient in which mathematics achieved earliest development was China, if we except India which lacks accurate data.

Mathematics, or the method of calculation which forms its foundation, as part of the advancement of mathematics, progressed by means of instruments in ancient China.

In this paper, we shall try to explain the varied circumstances under which since remote antiquity the methods of multiplication and division by means of instruments came to be developed. However, as to the earliest days, as in the history of other fields in China, the history starts from half-traditional accounts which are hardly accurate. Therefore, we shall proceed to investigate the development of the methods of multiplication and division based on reliable materials and some deductions, chronologically, from the Three-Dynasty Period 三國時代 (220~280 A. D.) when the method of calculation by means of instruments was definitely recorded to the last days of the Ming period (1368–1644) when the use of the abacus came to be extensively diffused.

2. Concerning *Sangi* さんぎ (Reckoning-blocks)

In China, if we except the earliest days with only halftraditional accounts, the earliest instrument used for calculation was *san-gi*. The reckoning blocks, it is admitted, were considerably used in the days of Chou 周 and Han 漢 (B. C. 1122~A. D. 220) which were feudalistic states.

As for the name *Sangi* Li Yen 李儼 in his *Chung-suan-shih-lun-ts'ung* 中算史論叢 (A Paper on the History of Chinese Mathematics) Pt. 4 entitled Chou-suan-chih-to-ka'o 籌算制度考 (Inquiry into *ch'ou* 籌 Reckoning blocks), says:

"Ancient people employed *ch'ou* for calculation. The block were originally

1) With the assistance of Seiichi TOYA, Lecturer, Akatsuki Gakuen, and Hisao SUZUKI, Lecturer, Dept. of Politics and Economics, Kokushikan University. The present article was first published in Japanese under the title of *Kigu ni yoru jōjo-keisan-hō no rekishi* 器具による乘除計算法の歴史, in *Shuzan Shichō* 珠算思潮, No. 1, Oct. 1960, the organ of the Japan Abacus Calculation Society. Refer to another article by the same author, *The Origin of the Chinese Abacus*, Memoirs of the Toyo Bunko, No. 18, 1960, which is itself an English version of *Chūgoku ni okeru soroban no kigen* 中國におけるそろばんの起源, the first chapter of *Shuzan-sampō no Rekishi* 珠算算法の歴史.

called *ts'ê* 策, came to be called *suan* 算, *ch'ou* 籌, *ch'ou-suan* 籌算 *ts'ê-suan* 策算, or *suan-ch'ou* 算籌, until they came to be commonly called *suan-tzŭ* 算子 (calculative pieces)."

Various ages had various names for them. In Japan, we commonly call them *san-gi* さんぎ.

Originally, the reckoning-blocks were in the form of slender columns, later they became square pillars. As for the materials, Li Yen *op. cit.* says:

"Originally, bamboo was used. This accounts for the inclusion of the character *chu* 竹 (bamboo) in the upper part of the ideographs *ts'ê*, *suan*, *chou*. Sometimes wood was used. It is for this reason that the *Fang-yen* 方言 explains *ts'ê* 策 as a slender branch. It also says, that later iron, ivory, and precious stone were used sometimes."

As for using blocks for numbers, the *Sung-tzŭ-suan-ching* 孫子算經 (a book written about the middle of the 3rd century) says:

"As the basis of calculation, one must master the method of representing numbers. For units, arrange blocks lengthwise, for tens sidewise, for hundreds vertical and for thousands horizontal. For thousands and tens, ten thousands and hundreds, arrange the blocks in the same way. For numbers over 6, put on top a block which represents 5: don't represent 6 by arranging 6 pieces in pallarel, while 5 is not represented by a single block."

Such later mathematic works as the *Yang-hui-suan-fa* 楊輝算法 (1274) and the *Hsiang-ming-suan-fa* 詳明算法 (1373) contain numbers represented by reckoning-blocks. 5 is represented as |||||, and 7 as ╥. If, in this case 31 is represented as ||| |, it will easily be taken to represent as |||| (4); therefore, it was decided that a number of several positions such as 7831, for example, should be represented as ⊥ 丌 三 |, that is, lengthwise and sidewise alternately.

By employing reckoning-blocks for numerals, it was now possible to operate various calculations.

3. Ancient Methods of Multiplication and Division

The oldest Chinese record of the methods of multiplication and division is the previously quoted *Sung-tzŭ-suan-ching*. The methods of calculation described in this book may be graphically shown as follows:

The method of multiplication: As for $81 \times 81 = 6561$

Top row	81
Middle row	
Bottom row	81

Put 81, the multiplicand in the top row, and 81 the multiplier in the bottom row. Move 81, the multiplier in the bottom row to the position of 8 the top figure of 81 multiplicand in the top row.

Multiply 8 the top figure of 81 the multiplic-

and by 81 the multiplier in the bottom row in the order:
8×8=64; 8×1=8; Put the product in the middle row.

Top row	81
Middle row	648
Bottom row	81

Strike off 8 in the multiplicand after calculation is completed. Move down by one position 81 the multiplier, multiply 1 the multiplicand by 81 the multiplier, and add the product to the figure in the middle row. 6561 the result is obtained. Finally, strike off 1 the multiplicand and 81 the multiplier.

Top row	1
Middle row	6561
Bottom row	81

The method of Division:

As for 6561÷9=729

Put 6561 the dividend in the middle row and 9 the divisor in the bottom row, move up 9 the divisor in the bottom row to the position of 6 the top figure of 6561 the divident. As 6 the dividend does not include 9 the divisor, 9 the divisor is moved down by one position.

Top row	
Middle row	6561
Bottom row	9

As 65 the dividend is divided by 9 the divisor, the quotient is 7; therefore, put 7 in the top row, and subtract from 65 the divident 63 the product of 7 the quotient and 9 the divisor.

Top row	7
Middle row	6561
Bottom row	9

9 the divisor is moved down by one position. As 26 the dividend is divided by 9 the divisor, the quotient is 2; therefore, put 2 in the top row, and subtract from 26 the divident 18 the product of this and 9 the divisor.

Top row	72
Middle row	261
Bottom row	9

9 the divisor is moved further down by another position. As 81 the dividend is divided by 9 the divisor, the quotient is 9; therefore, put this in the top row, and subtract from 81 the dividend from 81 the product of this and 9 the divisor.

Top row	729
Middle row	81
Bottom row	9

Strike off 9 the divisor. The quotient 729 is obtained in the top row.

Top row	729

In this way, the multiplicand, the product, and the multiplier (or the quotient, the dividend, and the divisor) were put in the three rows, top, middle, and bottom, and multiplication or division was operated.

However, it seems that in the case of such a simpler calculation as doubling or halving a number, in stead of applying the foregoing complicated method, the multiplicand was put down in the row and at once doubled or the dividend was at once halved.

The *Chiu-chang-suan-shu* 九章算術 (which in 263 Liu Hui 劉徽 annotated to render it the standard version) says:

"As for reducing fractions to the lowest term, when a number can be halved, it is halved; when a number cannot be halved the denominator and the numerator are arranged one above the other."

For example, in calculating

$26 \div 8 = 3\frac{1}{4}$ it seems that the following process was adopted.

Top row	
Middle row	二 丅
Bottom row	𝍧

→

Top row	|||
Middle row	||
Bottom row	𝍧

→

Top row	|||
Middle row	|
Bottom row	||||

$26 \div 8$ ⟶ $3\frac{2}{8}$ ⟶ $3\frac{1}{4}$

The *Hsia-hou-yang-suan-ching* 夏侯陽算經 (c. the 6th century) in operating calculation of an undecimal compound number, for example, in calculating 13 *tuan* 端 and 2 *chang* 丈 (1 tuan 端=5 chang 丈) in terms of *tuan*, says "Double the numer in the position of chang"; therefore, it seems that operation probably went some what like this:

— ||| =	→	— ||| ≣

13 *tuan* 2 *chang* ⟶ 13.4 *tuan*

Again, the same book taking up the matter of converting weights (1 *chin* 斤=16 *liang* 両), for example, that of converting *liang* into *chin*, in explaining 384÷16, says "halving the number four times."

Since, such explanations as "doubling a number" and "halving a number" occur in the *Wu-ts'ao-suan-shu* 五曹算術 (c. the 5th century) and also in the *Wu-ching-suan-shu* 五經算術 (c. the 6th century), it seems that such processes were adopted as simplified operations.

4. First Gleam of Simplified Calculation

As stated in the foregoing, it was customary in multiplication and division to calculate by arranging reckoning-blocks in the three rows, top, middle, and bottom; and only in special cases in which a number was doubled or halved, a simplified operation was adopted to change the multiplicand or the dividend at its own position. This simplified operation came to be extended even to an ordinary case in which the multiplier happened to be a single-figure number.

In the *Hsia-hou-yang-suan-ching* there occur in a question with a single-place multiplier such explanations as "Multiply it by 5 begining with the bottom figure" and "Multiply it by 7 begining with the top figure." For example, in the case of $612\times7=4284$, the multiplicand was probably multiplied by 7 from the top number down.

丅-||	612 the multiplicand is put down.
42-||	$6\times7=42$
527||	$1\times7=7$
4284	$2\times7=14$

In this way, multiplication by a single-place multiplier came to be operated much more easily than by arranging the numbers in the three rows. As the result, even in the case of multiplication by a more than two-place multiplier, this method was adopted when the multiplier could be solved into one-place factors. Calculation was made much more simply this way.

The reason why the *Hsia-hou-yang-suan-ching* offers "Multiply x first by 6 and then by 7" this explanation: on $x\times42$, and another: "Multiply x first by 3 and then by 8" on $x\times24$, is only because when a multiplier was a one-place figure the above mentioned calculation was adopted to conduct simpler and faster calculation.

Furthermore, even to a case in which the multiplier was a more than two-place number and could not be solved into one-place factors, this operation came to be extended. Among the questions for calculating taxes in the *Hsia-hou-yang-suan-ching*, for example, in calculating $428\times23=9844$, it explains; "Put down the multiplicand in two rows, multiply the upper figure by 2, and move down by one place the lower number and multiply it by 3, and add up the upper and lower figures."

428 428	Put down 428 the multiplicand in two rows, moving down by one place the lower figure.
856 1284	Multiply 428 the upper multiplicand by 2 and make it 856. Multiply 428 the lower multiplicand by 3 and make it 1284.
9844	Add 1284 the lower product to 856 the upper product, and the result 9844 will be obtained.

In this way, not only when the multiplier was a one-place number, but also when it was a two-place number or more, the multiplicand in its own position was changed in operating calculation.

5. Development of the *Shên-wai-chia* 身外加 method (addition after the figure itself) and *Shên-wai-chien* 身外減 method (subtraction after the figure itself)

Since ancient China was a feudal state founded upon the basis of agriculture, calculation of taxes formed one of important duties of the government officials. Consequently, Chinese mathematics developed as the mathematics for the government officials.

The *Hsia-hou-yang-suan-ching* contains a number of questions of calculating cereals to be paid in addition to the taxes, for example, one of calculating a tax of 2 additional bushels per 10 bushels.

For example, in calculating 428×12=5136, the above-mentioned method might be employed as follows:

(1)		(2)		(3)
428 428	⟶	428 856	⟶	5136
		428×2=856		4280+856=5136

In other words, it was known to be correct if to 428 the multiplicand, twice as much with one-place moved down was added. As the result, it is considered that this kind of calculation developed the following method.

𝍣𝍪𝍧	④②⑧	The multiplicand was put down.
	+ 1 6	8×2=16 was added.
𝍣𝍪𝍨𝍮	④② 9 6	
	+ 4	2×2=4 was added.
𝍣𝍫𝍢𝍮	④ 3 3 6	
	+ 8	4×2=8 was added.
𝍤𝍩𝍢𝍮	5 1 3 6	

The *Hsia-hou-yang-ching* for the calculation of A×102 directs "Add twice as much beyond one place"; for the calculation of A×16 directs "Add 6 times as much." These directions evidently refer to such calculations. That is, when the top figure of the multiplier happens to be 1, multiply the multicand by the multiplier, minus 1 the top number and add it to the multiplicand. Latter this came to be called the *Shên-wai-chia* method.

Again, the *Hsia-hou-yang-suan-ching* contains such a question of calculating taxes, ..for example, as for the cereals paid as a tax, if 2 bushels were allowed for carriage per 10 bushels paid as a tax, what was the net tax? The explanation for example reads: for calculating A÷12, "Take off twice as much," or

"Subtract twice as much from outside the quotient." It may be seen that, in this calculation,

$$1476 \div 12 = 123$$

with 1 the top number of the divisor stricken off, and considering only 2 the divisor, calculation was probably conducted in this way.

—||||≟T	1 4 7 6 −2	1476 the dividend is put down. Twice 1 the top figure of the dividend is subtracted.
—||≟T	①2 7 6 −4	Twice 2, the top figure of 276 the remaining dividend is subtracted.
—||三⊥	①②3 6 −6	Twice 3, the top figure of 36, the remaining divided is subtracted.
—||三	①②③	123 the result is obtained.

Like the method of the *Shên-wai-chia* method in multiplication, the division conducted through omitting the one top figure came to be called the *Shên-wai-chien* method. This method, compared with the ancient division separately conducted in the three rows, top, middle, and bottom, respectively representing the quotient, the dividend, and the divisor is more convenient as the quotient may be discovered more automatically.

For example, the calculation of 576÷18=32 is explained as follows:

Π̄| |||||≟T	8	5⑦⑥	The dividend and 8 the remainder of the divisor after subtracting 1, the top figure is put down. If 5 the top figure taken up as the quotient, the product 40 of 5 the quotient and 8 the divisor cannot be subtracted from 7 the following place number. Therefore, the quotient is decreased by 1, and 10 is returned to the next place.
	8	−1 +10	
Π̄| ||||≟T	8	4⑰⑥	32 the product of 4 the quotient and 8 the divisor cannot be subtracted from 17 the next place number; therefore, the quotient is further decreased by 1 and 10 is returned to the next place.

	−1	
	+10	
𝍧 |||≟T	8 3 ㉗ ⑥	
	−㉔	
𝍧 |||≡T	8 3 ③ ⑥	
	−1	
	+10	
𝍧 |||=−T	8 3 2 ⑯	
	−⑯	
𝍧 |||=	8 3 2	

24 the product of 3 the quotient and 8 the divisor is subtracted from 27 in the next place.

3 the top figure of 36 the remaining dividend is taken up as the quotient, 24 the product of 3 the quotient and 8 the divisor connot be subtracted from the next place number.

The quotient is decreased by 1, and 10 is returned to the next place. 16 the product of 2 the quotient and 8 divisor is subtracted from 16 in the next place.

32 the quotient is obtained.

By adopting the foregoing process, the quotient may be obtained without any thought of discovering the quotient. It was for this reason that the method came to play an important part in the T'ang period.

6. Diffusion of the *Ch'iu-i* 求一 method

Over against the ancient multiplication-division method of calculating by means of reckoning-blocks arranged in three rows, there arose several simplified calculative methods mentioned in the foregoing. The T'ang period saw a diffusion of a calculative method which extended the ideas of the *Shên-wai-chia*-method and the *Shên-wai-chien* method.

In multiplication,

432×24	was put down as	864×12
432×48	was put down as	1728×12
432×63	was put down as	216×126

Thus multiplicands or multipliers were doubled or halved, thereby the top figure of the multiplier was changed to 1 so that the *Shên-wai-chia* method might be applied.

In division,

756÷24	was put down as	378÷12
756÷48	was put down as	189÷12
756÷63	was put down as	1512÷126

Thus dividends or divisors were doubled or halved there-by the top figure of the divisor was changed to 1 so that the *Shên-wai-chien* method might be applied. This calculative method was called the *Ch'iu-i* method. And as this method appeared, the *Shên-wai-chia* method and the *Shên-wai-chien* method which had previously been utilized only in special calculation when multipliers or divisors had 1 as their top number, could now be applied to all cases. Consequently, it would seem that these methods were considerably employed during the T'ang period. For the *Yang-hui-suan-fa* 楊輝算法 (1274), a mathematics book of the Sung period rather thoroughly explains the *Ch'iu-i* method; on the other hand, a later book of the Yüan period entitled *Suan-hsüeh-ch'i-mêng* 算學啓蒙 (1299), asserting that the *Ch'iu-i* 求一 method is inferior to the *Kuei-ch'u* 歸除 method (which will be discussed later) omits the *Ch'iu-i-method.* The *Suan-fa-ch'üan-nêng-chi* 算法全能集 (c. the 14th century) states that, since the advent of the *Kuei-ch'u* method, the *Ch'iu-i* 求一 method is not used, though in the earlier days the method was considerably favoured, nowadays only those versed in ancient calculation method know this. This is what the book says.

From these records it may be admitted that the *Ch'iu-i* method was considerably employed prior to the Sung period when the *Kuei-ch'u* method came into being.

According to the Lü-li-chih 律暦志 of the *Sung-shih* 宋史 by Li Yen 李儼, "Ch'ên Ts'ung-yün 陳從運 of the T'ang period wrote the *Tê-i-suan-ching* 得一算經. His method of calculation is said to have consisted of multiplying a figure by a one-place number or by halving it, and then addition or substraction is applied." Since the *Tê-i-suan-ching* has been lost sight of, it is impossible to find out its contents. Judging from the contexts, it may be supposed that this book of the T'ang period also contained the *Ch'iu-i* method.

7. Advent of the Chiu-Kuei 九歸 Method of Division

While the calculative method of multiplication and division was practiced as the normal method from ancient times, we have seen in the foregoing section that the *Ch'iu-i* method appeared as a simplified method in the T'ang-period.

During the Sung period, in the field of division, there appeared a new method of seeking the quotient by means of the supplementary number of the divisor. This is a method of seeking the quotient without using the multiplication-table.

This is explained in the *Mêng-ch'i-pi-t'an* 夢溪筆談, which is not an arithmetic book, written by Ch'ên-ch'ieh 沈括 (B. C. 1930–94) of North Sung 北宋. The book says:

"The *Tsêng-ch'êng-i* 增成一 method, unlike other methods no use of multiplication or division. It is operated only by adding the supplementary number which is missing in the divisor. For example, if you desire to divide a number by 9, you will add 1, and if you desire to divide it by 8, you will add 2." As for, $272 \div 8 = 34$

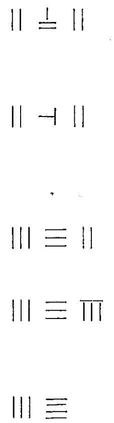
𝍡 𝍯 𝍡	2⑦② +2 +2	Seeing 2 the top number of the dividend, twice add 2 the supplementary number of 8 the divisor to the next place. The quotient will be 2.
𝍡 ⊢ 𝍡	2⑪② −8 +1	From 11 the dividend in the next place subtract 8 the divisor and add 1 to the quotient. The quotient will be 3.
𝍢 𝍫 𝍡	3③② + 2 + 2 + 2	Seeing 3 the top figure of 32 the remaining dividend, add 2 the supplementary number three times to the next place. The quotient will be 33.
𝍢 𝍫 𝍧	3 3⑧ − 8 + 1	From 8 the dividend, 8 the divisor is subtracted, and 1 is added to quotient. The quotient will be 34.
𝍢 𝍬	3 4	

It was probably operated in the above-mentioned manner. Before long, however, this method utilized the multiplication-table, and it was most probably changed as follows:

𝍡 𝍯 𝍡	2 ⑦ ② +4	2 the top figure of the dividend is multiplied by 2, the supplementary number of 8 the divisor, and 4 the product is added to the next place. The quotient will be 2.
𝍡 ⊢ 𝍡	2 ⑪ ② − 8 +1	From 11 the dividend in the next place, 8 the divisor is subtracted, and 1 is added to the quotient. The quotient will be 3.
𝍢 𝍫 𝍡	3 ③ ② + 6	3 the top figure of 32 the remaining dividend is multiplied by 2 the supplementary number of 8 the divisor, and 6 the product is added to the next place. The quotient will be 33.
𝍢 𝍫 𝍧	3 3 ⑧ − 8 + 1	From 8 the dividend, 8 the divisor is subtracted and 1 is added to the quotient. The quotient will be 34.
𝍢 𝍬	3 4	

As, in this method, not only is the quotient automatically sought, but also is calculation rendered more simplified, it must have been favored extensively.

Furthermore, as this method was repeatedly employed, it may be inferred, the people now developed which a system of chants by comparing by sight the top figure of the divisor and the top figure of thedividend enabled them to tell what would be the quotient. In other words, in the above-mentioned exercise:



2 ⑦ ②
+ 4
2 ⑪ ②
−8
+1
3 ③ ②
+ 6
3 3 ⑧
− 8
+ 1
3 4

As the top figure of the dividend is 2, chanting Ch'ien-êrh-hsia-ssu 見二下四 (See-two-add-four-down) add 4 to the next place. Chanting Fêng-pa-ch'êng-shih 逢八成十 (Meet-eight-make-ten), subtract 8 the divisor from 11, the dividend and advance 1 by one place and add 1 to the quotient.

As the top figure of the remaining dividend is 3, chanting Ch'ien-san-hsia-liu 見三下六 (See-three-add-six-below), add 6 to the next place.

Chanting Fêng-pa-ch'êng-ten 逢八成十 (Meet-eight-make-ten), subtract 8 the divisor from 8 the dividend, advance 1 by one place and add 1 to the quotient. 34 the quotient will be obtained.

Though these chants were no doubt somewhat variegated in different periods, in the course of time they came to assume the following forms. They are so formed that by seeing the top figure of the divisor and the top figure of the dividend, the quotient and the remainder may be indicated.

Division by 2:—

Êrh-i-t'ien-tso-wu	二一添作五	(Two-one-attach-make-five)
F'êng-êrh-chin-ch,ngê-shih	逢二進成十	(Meet-two-above-make ten)

Division by 3:—

San-i-san-shih-i	三一三十一	(Three-one-make-thirty-one)
San-êrh-liu-shih-êrh	三二六十二	(Three-two-make-sixty-two)
Fêng-san-chin-ch'êng-shih	逢三進成十	(Meet-three-above-make-ten)

Division by 4:—

Ssŭ-i-êrh-shih-êrh	四一二十二	(Four-one-make-twenty-two)

Ssŭ-êrh-t'ien-tso-wu	四二添作五	(Four-two-attach-make-five)
Ssŭ-san-ch'i-shih-êrh	四三七十二	(Four-three-make-seventy-two)
Fêng-ssŭ-chin-ch'êng-shih	逢四進成十	(Meet-four-above-make-ten)
Division by 5:—		
Wu-i-pei-tso-êrh	五一倍作二	(Five-one-double-make-two)
Wu-êrh-pei-tso-ssŭ	五二倍作四	(Five-two-double-make-four)
Wu-san-pei-tso-liu	五三倍作六	(Five-three-double-make-six)
Wu-ssŭ-pei-tso-pa	五四倍作八	(Five-four-double-make-eight)
Fêng-wu-chin-ch'êng-shih	逢五進成十	(Meet-five-above-make-ten)
Division by six:—		
Liu-i-hsia-chia-ssŭ	六一下加四	(Six-one-add-four-below)
Liu-êrh-san-shih-êrh	六二三十二	(Six-two-make-thirty-two)
Liu-san-t'ien-tso-wu	六三添作五	(Six-three-attach-make-five)
Liu-ssŭ-liu-shih-ssŭ	六四六十四	(Six-four-make-sixty-four)
Liu-wu-pa-shih-êrh	六五八十二	(Six-five-make-eighty-two)
Fêng-liu-chin-ch'êng-shih	逢六進成十	(Meet-six-above-make-ten)
Division by 7:—		
Ch'i-i-hsia-chia-san	七一下加三	(Seven-one-add-three-below)
Ch'i-êrh-hsia-chia-liu	七二下加六	(Seven-two-add-six-below)
Ch'i-san-ssŭ-shih-êrh	七三四十二	(Seven-three-make-forty-two)
Ch'i-ssŭ-wu-shih-wu	七四五十五	(Seven-four-make-fifty-five)
Ch'i-wu-ch'i-ch'i-shih-i	七五七十一	(Seven-five-make-seventy-one)
Ch'i-liu-pa-shih-ssŭ	七六八十四	(Seven-six-make-eighty-four)
Fêng-ch'i-chin-ch'êng-shih	逢七進成十	(Meet-seven-above-make-ten)
Division by 8:—		
Pa-i-hsia-chia-êrh	八一加下二	(Eight-one-add-two-below)
Pa-êrh-hsia-chia-ssŭ	八二加下四	(Eight-two-add-four-below)
Pa-san-hsia-chia-liu	八三加下六	(Eight-three-add-six-below)
Pa-ssŭ-t'ien-tso-wu	八四添作五	(Eight-four-attach-make-five)
Pa-wu-liu-shih-êrh	八五六十二	(Eight-five-make-sixty-two)
Pa-liu-ch'i-shih-ssŭ	八六七十四	(Eight-six-make-seventy-four)
Pa-ch'i-pa-shih-liu	八七八十六	(Eight-seven-make-eighty-six)
Fêng-pa-chin-ch'êng-shih	逢八進成十	(Meet-eight-above-make-ten)
Division by 9:—		
Chiu-i-hsia-chia-i	九一下加一	(Nine-one-add-one-below)
Chiu-êrh-hsia-chia-êrh	九二下加二	(Nine-two-add-two-below)
Chiu-san-hsia-chia-san	九三下加三	(Nine-three-add-three-below)
Chiu-ssŭ-hsia-chia-ssŭ	九四下加四	(Nine-four-add-four-below)
Chiu-wu-hsia-chia-wu	九五下加五	(Nine-five-add-five-below)
Chiu-liu-hsia-chia-liu	九六下加六	(Nine-six-add-six-below)

Chiu-ch'i-hsia-chia-ch'i	九七下加七	(Nine-seven-add-seven-below)
Chiu-pa-hsia-chia-pa	九八下加八	(Nine-eight-add-eight-below)
Fêng-chiu-chin-ch'êng-shih	逢九進成十	(Meet-nine-above-make-ten)

The method of division operaten by means of these chants is called the *Chiu-kuei* 九歸 method.

The exact date of the advent of the Chiu-kuei method is not known, but seeing that the *Yang-hui-suan* method 楊輝算法 (1274) says that "The *Chi-nan-suan-fa* 指南算法 describes the *Shên-wai-chia* method the *Shên-wai-chien* method the *Chiu-kuei* method and the *Ch'iu-i* method. From this it is evident that in 1078–1189 when the *Chi-nan-suan-fa* was written the *Chiu-kuei* method had already existed. However, the *Chi-nan-suan-fa* itself is not extant.

After the advent of the *Chiu-kuei* method an attempt was made to extend this method to division by more than two place divisors.

The *Yang-hui-suan-fa* says: "In the division by one-place-devisor, instead of the *Shang-ch'u* 商除 method (the ancient division operated by arranging the figures in 3 rows the top, middle, and the bottom), the *Chiu-kuei* method is employed. In the division by two-place or three-place devisors, the *Shang-ch'u* method is more suitable. People nowadays employ the *Chiu-kuei* method in an extended form; referring to the top figure of the devisor, they seek the quotient by reciting the *Chiu-kuei* chants and adopt the method of subtracting from the dividend the product obtained by successively multiplying the quotient and the second-place and third-place numbers of the divisor. In other words, they apply both the *Chiu-kuei* method and the Shang-ch'u method in one and the same calculation." In this method, for example, in calculating 684÷38, seeing that the top figure of the divisor is 3, they use the chant of divison by 3.

38	⑥ ⑧ ④	Chanting Fêng-san-chin-ch'êng-shih 逢三進成十 (Meet-three-above-make-ten), they subtract 3 from 6 the dividend, advance 1 to the place above it to make 1 the quotient.
	−3	
	+1 0	
38	1 ③ ⑧ ④	8 the product of 1 the quotient and 8 the second place figure of the divisor is subtracted at the second place below from the quotient.
	− 8	
38	1 ② ⑩ ④	
38	1 6 ⑫ ④	Chanting San-êrh-liu-shih-êrh 三二六十二 (Meet-three-two-make-sixty-two), 2 the top figure of ② ⑩ ④ the remaining is changed to 6, and 2 is added to the next place.
	− 3	
	+ 1 0	
	− 3	
	+ 1 0	Chanting twice Fêng-san-chin-ch'êng-shih 逢三進成十 (Meet-three-above-make-ten), from 12 the dividend 3 is subtracted twice, and 1 is advanced twice to a higher place, the quotient is 8.
38	1 8 ⑥ ④	
	− 6 4	
38	1 8	

The product of 8 the quotient and 8 the second-place number of the divisor is subtracted from 64, 18 the quotient is obtained.

In the case of dividing 2301 by 39 in this method, considerable thinking is needed. Therefore, it would seem that in the division by more-than-two place divisors the *Chiu-kuei* method was not used frequently: In and after the Yüan period, however, combined with such new methods as the *Huan-yüan* 還原 method and the *Ch'ung-kuei* 撞歸 method adopted, this method, under the new name the *Kuei-ch'u* 歸除 method, for the first time began to prove its real value as an exceedingly simplified division method and enjoy an extensive popularity.

8. Progress of Multiplication

The methods of division practised during the Sung period were:

(1) The *Shang-ch'u* method handed down from ancient times in which calculation were operated by means of the reckoning-blocks arranged in the three rows, the top, the middle, and the bottom.

(2) The *Shên-wai-chien* 身外減 method as a simplified method used in cases where the top figure of the divisor is 1, and the *Ch'iu-i* 求一 method which is only a practical application of the same;

(3) And the *Chiu-kuei* method which at length came into being.

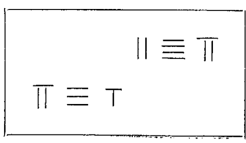
Now, how fared the method of multiplication? The *Yang-hui-suan-fa* explains: When the divisor is of a single place, "First memorize the divisor, start multiplying the multiplicand beginning with the numbers in the higher places; where the multiplication table includes "make ten", put down the product beginning with one place above the multiplicand, and where the chant contains a blank place change the multiplicand in each chant for the product." It is evident that the method practised since the North-South dynasty periods was still being operated in entirety.

When the divisor is of more than two place figures, they were called *Hsiang-ch'êng* 相乘 (mutual multiplication).

In calculating:

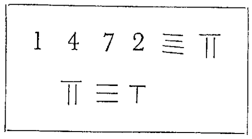
$$247 \times 736 = 181792$$

the following diagram is given:

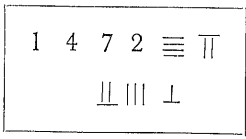


247 the multiplicand and 736 the multiplier are put down in the top and bottom rows so that 2 the top figure of the multiplicand and 6 the bottom figure of the multiplier come into the same place.

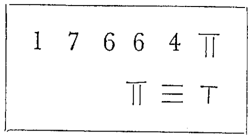
2 the top figure of the multiplicant and 736 the multiplier are multiplied (2×7, 2×3, 2×6 successively) and obtain 1472.



736 the multiplier is lowered by one place, and is multiplied by 4 the multiplicand; and 2944 the product is added to 14720 and 17664 is obtained.



736 the multiplier is further lowered by one place and is multiplied by 7; and 5152 the product is added to 17664 and 181792 is obtained.



1	8	1	7	9	2

In this way, during the Sung period the multiplication method operated by arranging reckoning-blocks in the three horizontal rows, the top, the middle, and the bottom, came to be simplified to be operated in the two horizontal rows, the top and the bottom. Besides, this multiplication method was operated as the ordinary method, and the *Shên-wai-chia* 身外加 method used as a simplified method when the top figure of the multiplier is 1, and also its practical application called the *Ch'iu-i* method were practised.

9. The Perfection of the *Kuei-ch'u* Method

Over against the ancient *Shang-ch'u* method of calculating by means of arranging reckoning-blocks in three horizontal rows, top, middle and bottom; the *Ch'iu-i* 求一 method came to be employed both for a greater ease of seeking the qnotient and for a greater simplicity of the calculative method, while as has been mentioned there came into being the *Chiu-kuei* method, ..a method of division operated by means of special chants for automatically seeking the quotient.

The *Chiu-kuei* method which was at first employed only when the divisor happened to be a one-place number, came to be used also when the divisor was of a more-than-two-place number. In the latter cases, however, it seems that it was not usually employed because of difficult calculations involved.

However, as (1) The *Huan-yüan* method for revising the quotient in which, as in the *Shên-wai-chien* method, when the quotient is too great, calculation is operated by subtracting 1 from the quotient and by adding to the dividend the top figure of the divisor, and

(2) The *Ch'ung-kuei* method in which when the top place of the dividend and the divisor happen to be of the same number and the number in the next

place of the dividend happens to be smaller (as, for example, 3024÷36), reciting the chant "Chien-san-wu-ch'u-tso-chiu-san 見三無除作九三 (See-three-no-dividend-make-ninety-three), 9 is temporarily considered the quotient, as these two methods came to be used concurrently, even when the divisor was of a more-than-two-place number, calculation could be operated most easily and quickly. As to the chants of the *Chiu-kuei* method, were added the chants of the *Huan-yüan* 還原 method and of the *Ch'ung-kuei* 撞歸 method which are given in the following, the method of division called *Kuei-ch'u* method for division with a more than twoplace divisor was now perfected.

Division by 2:—

Chien-êrh-wu-ch'u-tso-chiu-êrh 見二無除作九二 (See-two-no-dividend-make-ninety-two)

Wu-ch'u-chien-i-hsia-huan-êrh 無除減一下還二 (No-dividend-subtract-one-add-return-two-below)

Division by 3:—

Chien-san-wu-ch'u-tso-chiu-san 見三無除作九三 (See-three-no-dividend-make-ninety-three)

Wu-ch'u-chien-i-hsia-huan-san 無除減一下還三 (No-dividend-subtract-one-add-return-three-below)

Division by 4:—

Chien-ssŭ-wu-ch'u-tso-chiu-ssŭ 見四無除作九四 (See-four-no-dividend-make-ninety-four)

Wu-ch'u-chien-i-hsia-huan-ssŭ 無除減一下還四 (No-dividend-subtract-one-add-return-four-below)

Division by 5:—

Chien-wu-wu-ch'u-tso-chiu-wu 見五無除作九五 (See-five-no-dividend-make-ninety-five)

Wu-ch'u-chien-i-hsia-huan-wu 無除減一下還五 (No-dividend-subtract-one-add-return-five-below)

Division by 6:—

Chien-liu-wu-ch'u-tso-chiu-liu 見六無除作九六 (See-six-no-dividend-make-ninety-six)

Wu-ch'u-chien-i-hsia-huan-liu 無除減一下還六 (No-dividend-subtract-one-add-return-six-below)

Division by 7:—

Chien-chi-wu-ch'u-tso-chiu-chi 見七無除作九七 (See-seven-no-dividend-make-ninety-seven)

Wu-ch'u-chien-i-hsia-huan-chi 無除減一下還七 (No-dividend-subtract-one-add-return-seven-below)

Division by 8:—

Chien-pa-wu-ch'u-tso-chiu-pa 見八無除作九八 (See-eight-no-divinded-make-ninety-eight)

Wu-ch'u-chien-i-hsia-huan-pa 無除減一下還八 (No-dividend-subtract-one-add-return-eight-below)

Division by 9:—

Chien-chiu-wu-ch'u-tso-chiu-chiu 見九無除作九九 (See-nine-no-dividend-make-ninety-nine)

Wu-ch'u-chien-i-hsia-huan-chiu 無除減一下還九 (No dividend-subtract-one-and-return-nine-below.)

Here is a calculation exercise in which the chants of the *Huan-yüan* and *Ch'ung-kuei* methods may be used. 2301÷39=59

39	② ③ ⓪ ①	Looking at 3 the divisor and 2 the top figure of the dividend, recite San-êrh-liu-shih-êrh 三二六十二 (Three-two-make-sixty-two), change 2 to 6 and add 2 to the next place.
	6 ⑤ ⓪ ①	Now that the quotient has become 6, 54 the product obtained by multiplying it by 9, the figure in the next place of the divisor is subtracted. As 54 cannot be subtracted from 50, reciting Wu-ch'u-chien-i-hsia-huan-san 無除減一下還三 (No-dividend-subtract-one-add-return-three below), subtract 1 from 6 the quotient, and add, to the next place of the quotient 3 the top figure of the divisor.
	−1	
	+3	
39	5 ⑧ ⓪ ①	45 the product of 5 the quotient multiplied by 9 the next figure of the divisor is subtracted from 80 in the next places.
	− 4 5	
	5 ③ ⑤ ①	The top figure of 351 the remainder of the dividend is 3 and identical with the top figure of the divisor; but when the next places of both are compared, that of the dividend is the smaller, reciting Chien-san-wu-ch'u-tso-chiu-san 見三無除九三 (See-three-no-dividend-make ninety-three), change 3 to 9, and add 3 to the next place.
	+6	
	+3	
	5 9 ⑧ ①	Now that 9 has become the quotient, 81 the product multiplied by 9 the figure in the next place of the divisor is subtracted from 81 in the next places.
	− 8 1	
39	5 9	59 the quotient is obtained.

It is not known exactly when the *Huan-yüan* method and the *Ch'ung-kuei* method were adopted by the *Chiu-kuei* method, and the perfect method of division called the *Kuei-ch'u* method. However, the *Suan-hsüeh-ch'i-mêng* 算學啓蒙 (1299) by Chu Shih-chieh 朱世傑 of the Yüan period, in connection with the *Chiu-kuei-ch'u* 九歸除 method, says "When the dividend is smaller than the divisor, according to the chants of the *Chiu-kuei* method, change the top figure of the dividend for the quotient, when the dividend is greater than the divisor, the quotient is carried beyond the dividend. Care must be taken in this case to retain within the dividend the product of the quotient multiplied by the figures below the second place of the divisor.

"When the top figure of the divisor and the dividend are similar,make it ninety odds."

"If, in seeking the quotient, the product of the quotient multiplised by the second place figure of the divisor cannot be subtracted, reduce 1 from the quotient and restore the top figure of the divisor." From the existence of some chants to this effect, it may be infered that by this time the *Chiu-kuei* method had already adopted the *Huan-yüan* method and the *Ch'ung-kuei* method.

The book further says: "Foremerly people commonly used the *Shang-ch'u* method. As this method is too difficult for beginners, later people came to use the *Kuei-chu* method in stead of the *Shang-ch'u* method. Yet this is not the normal method."

In spite of this fact, the book neglects to explain the *Shang-ch'u* method, but explains the *Kuei-ch'u* method exclusively. Therefore, it may be said that the traditional *Shang-ch'u* method operated by arranging reckoning-blocks in three horizontal rows though recognized as the ordinary method was no more used, and the *Kuei-ch'u* method came to take its place.

Besides, the *Suan-fa-ch'üan-nêng-chi* of the 14th century says: "Since the completion of the *Kuei-ch'u* method, the *Ch'iu-i* 求一 method has never been used." "The *Shang-ch'u* method includes in one all the three methods, the *Chiu-kuei*, the *Shên-wai-chien*, the *Kuei-ch'u*, but is inferior to them in the speed of calculations."

The *Hsiang-ming-suan-fa* 詳明算法 (1373) says: "Though the *Shang-ch'u* method is a culculative method which includes in one all the *Chiu-kuei*, the *Shên-wai-chien*, and the *Kuei-ch'u*, a man need not learn it if he is versed in the *Kuei-ch'u* method. Only in calculating evolutions the *Shang-ch'u* method is invariably employed."

As these books state, since the coming of the *Kuei-ch'u* method came to be used, the *Ch'iu-i* method which had been extensively used since the T'ang period, and the *Shang-ch'u* method practised since remote antiquity, all these had gone out of use.

This being the case, division in these days, was operated separately as follows: When the divisor was a one-place number, the *Chiu-kuei* method was employed: and when the divisor was a more than two-place number, the *Shên-wai-chien* method was employed if the top figure of the divisor was 1, and the *Kuei-ch'u* method if the top figure of the divisor was 2～9.

10. The Perfection of the Method of Multiplication

While the method of division evolved from the ancient *Shang-ch'u* method and the *Ch'iu-i* method to the *Kuei-ch'u* method, the method of multiplication also underwent considerable changes.

The ancient method of multiplication operated by means of reckoning-blocks arranged in three horizontal rows was superceded in the Sung period by another operated by means of reckoning-blocks arranged in two horizontal rows; then in the Yüan period when division by means of the *Kuei-ch'u* method was extensively diffused, a new multiplication named *Liu-t'ou-cn'êng* 留頭乘 method came into being.

The *Suan-hsüeh-ch'i-mêng* 算學啓蒙 (1299) of the Yüan period says: "The *Liu-t'ou-ch'êng* method differs from the previous methods. To begin with, multiplication is started with the numbers at the second place of the multiplier; the number in the Chinese chant containing the character *shih* 十 (ten) in the multiplication table is put down after the multiplicand and the number in the Chinese chant containing the character *ju* 如 is put down with one place left blank. After the numbers below the second place number of the mutiplier are multiplied, come back to the top number and change the multiplicand, for the product."

This is the chants to the effect. Let us explain:

$$27\times236=6372$$

236	2	⑦				................the numbers are put down.
			2	1		............7×3
				4	2	..........7×6
		1	4			..............7×2, 7 the multiplicand is changed for 1
236	②	1	6	5	2	and 4 is added to the next.
		0	6			2×3
			1	2		2×6
	0	4				2×2, 2 the multiplicand is taken away and
		6	3	7	2	4 is added to the next one.

When the multiplier is a one-place number, it was called *Yin* 因 method (method of simplification) and as in the previous days, a multiplication process was under-taken beginning with the figures in the top place of the multiplicand, but since the period in which the *Hsiang-ming-suan-fa* (1373) was written the rule was revised to calculate from the bottom figure of the multiplicand.

In this way was now established the principle to start calculation in division from the left of the dividend and in multiplication from the right of the multiplicand. In multiplication calculation from the top place was now changed to calculation from the bottom place. What prompted this radical change cannot be known, but from the fact that this change coincides with the evolution of the *Kuei-ch'u* method, it may be attributed to a very simple idea that as in division the quotient comes to the left of the dividend, in multiplication the product should come to the right of the multiplicand.

Incidentally, the *Hsiang-ming-suan-fa* contains another method of multiplication other than the *Liu-t'ou-ch'êng* 留頭乘 method. For example, 27×236 is calculated as follows:

236		2	⑦				
				4	2	....7×6	
			2	1		.......7×3	
		1	4			.........7×2	7 the multiplicand is changed to 1 and 4 is added to the next place.
	②	1	6	5	2		
			1	2		2×6	
		0	6			2×3	
	0	4				2×2	2 the multiplicand is taken, and 4 is added to the next place.
		6	3	7	2		

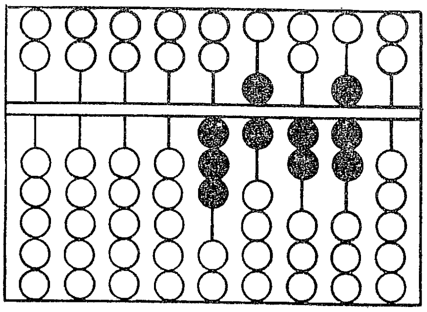
This is a method later called *Chao-i-ch'êng* 掉尾乘 method and considerably used in the Yedo 江戸 period in Japan. But in the Ming period, it seems, the *Liu-t'ou-ch'êng* method was generally used and this method was little used.

Therefore, multiplication in this period was operated separately as follows:

When the multiplier was a one-place number, the *Yin-method* was employed, and the multiplier was a more-than-two-place number, the *Shên-wai-chia* method was employed if the top figure of the multiplier was 1, and the *Liu-t'ou-ch'êng* method if the top place number was 2~9.

11. From the Reckoning-blocks to the Abacus

The calculative method by means of reckoning-blocks, while fully displaying its characteristics [and undergoing several improvements, since remote antiquity, had achieved a brilliant and unrivalled development by the middle of the Ming period.



3627 Represented on the Chinese Style Abacus.

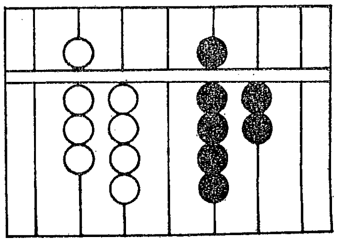
We may now turn to the abacus, a calculative instrument extensively employed even now in the Orient.

Mathematics in China had always been developed as an important science for the government officials. The reckoning-blocks had been employed chiefly by taxation-officers and mathematicians. On the other hand, the abacus seems to have been popular among the masses, the less intellectual tradesmen.

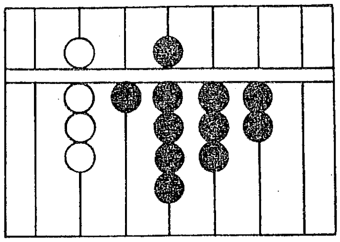
Consequently, it would seem that multiplication and division by means of the abacus were operated most crudely, multiplication being simply a repetition of additions and division simply a repetition of subtractions.

For example:

$84 \times 23 = 1932$

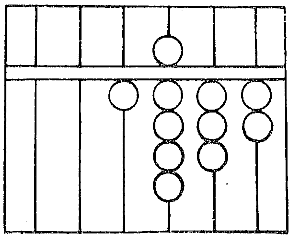


(1) 84 the multiplicand is put down, 23 the multiplier is added 4 times to obtain 92.

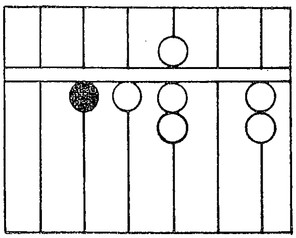


(2) 23 the multiplier is added 8 times to obtain 1932 the result.

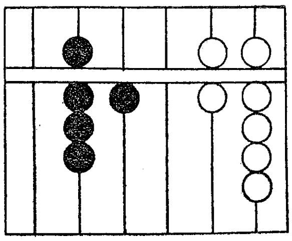
$1932 \div 23 = 84$



(1) 1932 the dividend is put down.



(2) Subtract 23 the divisor from the top place and add 1 for the quotient. Repeat this and obtain 8 the quotient and 92 the remainder.



(3) Repeat the same thing with 92 the remainder. 84 is the quotient with no remainder.

It would seem that the calculation was most probably operated like this. The abacus with only such crude methods of multiplication and division (probably so much handier for those unintellectual people) proved exceedingly convenient in addition or subtraction.

By the middle of the Ming period when trade arose and prospered in the basins of the Yang-tse River, the abacus seems to have been recognized and considerably diffused. In addition to this, it was most fortunate for the abacus that about this time the multiplication and division methods by means of reckoning-blocks had been changed (by adopting the *Liu-t'ou-ch'êng* 留頭乘 method and *Kuei-ch'u* method) so that they could very well be operated entire on the abacus.

Consequently, the abacus, by adopting the multiplication and division methods of reckoning-blocks, succeeded in achieving a far greater success than that of the reckoning-blocks.

The abacus which could now operate addition, subtraction, multiplication, division, evolution, by the end of the Ming period, had been diffused through the whole land. Having expelled the reckoning-blocks, it moved out into the lime-light as the only calculative instrument.

The abacus was introduced to Japan toward the end of the 16th century. The Japanese-style abacus which had improved the Chidese-style has achieved a success even more brilliant than that of the Chinese.

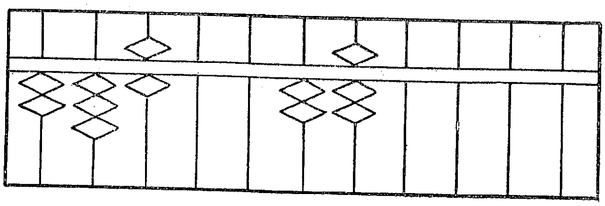
The abacus, the simplest, the cheapest, and the most damage-proof abacus, which is even today loved as a convenient calculative instrument in the Orient, has become a worthy support in our economic life.

For Reference:

In modern Japan, we use the abacus with somewhat angular beads, on columns which have each one bead representing five, and 4 others which represent one each, and is convenient for carrying about. Multiplication and division are operated in the following way. Though an expert can calculate squares root and cube, root and quadratic or cubic equation. However, such calculations will be omitted here.

Multiplication:

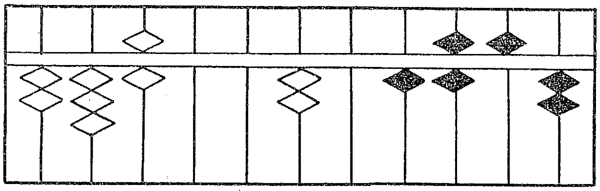
2 7 × 2 3 6



2 3 6 2 7

27 the multiplicand is put down at the centre and 236 the multiplier is put down on the left.

7×2..........			1	4			
7×3...............				2	1		
7×6...................					4	2	
		− 7					
	2	0	1	6	5	2	

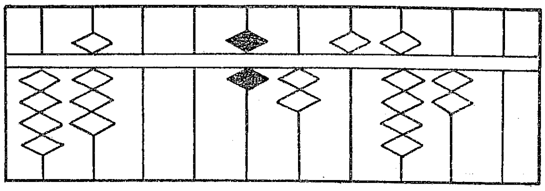


2 3 6 ② 0 1 6 5 2

2×2........	0	4			
2×3............		0	6		
2×6................			1	2	
− 2					
	6	3	7	2	

Division:

2592÷48



4 8 ⑥ 2 5 9 2

6×4=			− 2	4			
		⑥	0	1	9	2	
	− 1						
			+	4			
		⑤	0	5	9	2	
5×8=			−	4	0		
4 8		⑤	0	1	9		

The dividend is put down at the centre, and the divisor on the left.

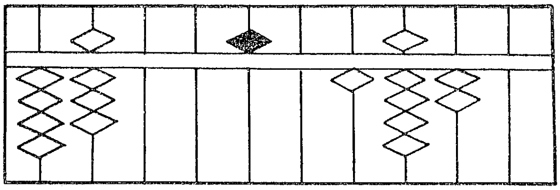
From 25÷4, the quotient is obtained as 6, and this is put up ahead.

6×4=24

This is subtracted from the dividend.

6×8=48

This cannot be subtracted from 19 the remainning dividend. The quotient is decreased by 1; and 4 the first figure of the divisor. is added to the dividend the quatient is 5. 5×8=40 This is subtracted from the remaining dividend.



4 8 ⑤ ④ 1 9 2

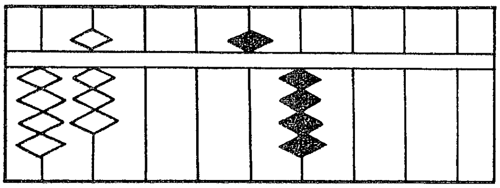
From $19 \div 4$, the quotient is obtained as 4.

$4 \times 4 = -1\ \ 6$

$4 \times 8 = \ \ -3\ \ 2$

⑤ ④

$4 \times 4 = 16$, $4 \times 8 = 32$
These are subtracted from the remainder of the dividend.



4 8 5 4